JIBAS

Journal of Insights in Basic and Applied Sciences

مجلة رؤى في العلوم الأساسية والتطبيقية

Volume 2, Issue 2, 2026, Pages: 119-130

Journal homepage: https://jibas.com.ly/index.php/jibas/en

Chaos Detection in the Logistic Map Using Lyapunov Exponents

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الكشف عن الفوضى في الخريطة اللوجستية باستعمال أسس ليابونوف

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Received: March 20, 2026 | **Accepted: April 25, 2026** | **Published: May 11, 2026**

Abstract:

This paper investigates the transition to chaos in the logistic map, a fundamental model in nonlinear dynamics, by employing Lyapunov exponents as a quantitative measure. We begin by establishing the mathematical framework of one-dimensional maps and the formal definition of the Lyapunov exponent as an indicator of sensitive dependence on initial conditions. A numerical method for computing the exponent is implemented and applied to the logistic map across a range of its control parameter. The resulting Lyapunov spectrum is then systematically compared to the map's bifurcation diagram, demonstrating how positive Lyapunov exponents precisely correspond to chaotic regimes. The analysis highlights the exponent's utility not only as a binary chaos indicator but also as a measure of chaos intensity. This study reaffirms the Lyapunov exponent as a robust and essential tool for characterizing dynamical systems.

Keywords: Lyapunov Exponent, Logistic Map, Chaos, Bifurcation, Sensitive Dependence on Initial Conditions, Nonlinear Dynamics, Finite-Time Lyapunov Exponents.

الملخص

تقدم هذه الورقة بحثاً عن الانتقال نحو الفوضى في الخريطة اللوجستية، والتي تعد نموذجاً أساسياً في الديناميكيات اللاخطية، وذلك باستخدام أسس ليابونوف كمقياس كمي. يوضع الإطار الرياضي للخرائط أحادية البعد والتعريف الرسمي لأس ليابونوف كمؤشر على الحساسية للشروط الابتدائية في البداية. لقد تم تنفيذ طريقة عددية لحساب الأس وتطبيقها على الخريطة اللوجستية على مجال محد لمتغير التحكم الخاص بها. بعد ذلك، تتم مقارنة طيف ليابونوف الناتج بشكل منهجي مع مخطط الانشعاب للخريطة، مما يوضح كيف تتوافق أسس ليابونوف الموجبة بدقة مع الأنظمة الفوضوية. تبين نتائج التحليل أن فائدة الأس ليست استخدامه فقط كمؤشر ثنائي للفوضى، بل أيضاً كمقياس لشدة الفوضى. تؤكد هذه الدراسة مجدداً على أن أس ليابونوف يعد أداة متينة وأساسية لتوصيف الأنظمة الديناميكية.

الكلمات المفتاحية: أس ليابونوف، الخريطة اللوجستية، الفوضى، الانشعاب، الحساسية للشروط الابتدائية، الديناميكيات اللاخطية، أسس ليابونوف ذات الزمن المنتهي.

Introduction

The ability to predict physical and biological systems has been a major concern for scientists, as prediction increases the ability to control and maintain these systems and ensure their future functioning. It has been believed that knowing the governing equations of a system with high accuracy may guarantee the prediction of its future behavior. As a result of the great development in nonlinear dynamics, it has been shown that simple nonlinear systems, which do not contain any randomness within their variables or states, can exhibit highly complex and unpredictable behavioral patterns. The mathematical concept underlying deterministic chaos was formulated by the meteorologist Edward Lorenz, which is the property of sensitive dependence on initial conditions, and this concept was called the butterfly effect [4]. The concept of the butterfly effect expresses that the presence of any very small and unobservable difference in the initial conditions increases at an exponential growth rate over time, leading this system to unpredictability in the long run [6].

The Logistic map is one of the most important mathematical tools used to study the dynamic behavior of systems in general, as it represents a simple mathematical model in terms of structure and capable of producing complex dynamic behavior. It is defined by the following one-dimensional iterative equation:

$$x_{n+1} = r \cdot x_n (1 - x_n) \tag{1}$$

where:

- r: Control parameter,
- x_n : The state variable at discrete time step n, which keeps the system finite so that it belongs to the domain [0,1].

The logistic map shows a large and varied area in the dynamic behavior of the system studied according to the value of the control parameter r. The map begins at a fixed stable point and then undergoes a series of period-doubling bifurcations occurring at certain values, culminating in fully chaotic behavior [3]. Feigenbaum [5] has demonstrated that this gradual transition to chaos has universal quantitative properties that apply to a wide class of one-dimensional maps.

Researchers relied on the visual examination of time series or bifurcation diagrams to determine the nature of the system, as these methods offer an explanation that provides an excellent degree of geometric understanding. But on the other hand, these tools are considered qualitative tools and are limited within the limits of the ability to distinguish visually, where in the case of transitional regions or when there is a periodic window within the chaotic range, it becomes very difficult and almost impossible to discern visually, which puts the burden on the researcher to identify the chaos within the system. This has led to the need for a mathematical tool that yields a single quantitative value that expresses the degree of chaos, to objectively quantify the degree of chaos in order to overcome visual constraints [7].

Based on the above, this research aims to use the Lyapunov exponent as a quantitative tool for exploring and characterizing chaos in the logistic map, which was developed by Alexander Lyapunov in the late 19th century as a tool for the study of stability, and today has become the most reliable measure for detecting chaos [1, 2]. To achieve this goal, this research will answer the following research questions:

1 How can the Lyapunov exponent be derived and numerically calculated for one-dimensional maps?
2 How does the value of the Lyapunov exponent change when the control parameter r changes across the range [2.5,4]?
3 How does the resulting spectrum of the Lyapunov exponent relate to the bifurcation diagram to accurately identify stable, cyclical, and chaotic regions?

This research is divided into six sections, where in addition to this first section, which provides a general introduction to the research and clarifies its objectives, tools, and methodology, the second section introduces the mathematical theoretical framework of the logistic map and explains the precise definition of the Lyapunov exponent. The third section is devoted to describing the numerical methodology and software algorithm used in calculation. The fourth section presents and analyzes the results visually and numerically by comparing the Lyapunov spectrum with the bifurcation diagram. Section V discusses the results and explains the strength of the tool used with reference to numerical constraints. Finally, Section VI summarizes the research findings.

2 Material and methods

2.1 Theoretical Framework

The transition will be made from qualitative description to a precise quantitative characterization of dynamical phenomena.

2.1.1 Logistics Map Dynamics

The logistic map is defined as a one-dimensional nonlinear dynamical system and can be represented mathematically by the following discrete-time equation [3]:

$$x_{n+1} = f_r(x_n) = r \cdot x_n (1 - x_n) \tag{2}$$

where $r \; [0,4]$ is the control parameter, and $x \in [0,1]$ is the state variable at discrete time step n.

The derivative of the function f with respect to the state variable x_n is calculated in order to analyze the stability of the system, where the derivative is given as follows:

$$f_r'(x_n) = r \cdot (1 - 2x_n) \tag{3}$$

A fixed point, denoted as (x^*), is defined as the point that satisfies the following relationship:

$$f_r(x^*) = x^* \tag{4}$$

The fixed point is stable if the following condition is met:

$$\left| f_r'(x^*) \right| < 1 \tag{5}$$

When the control parameter is increased, the system begins to lose its stability via a series of period-doubling bifurcations, which Feigenbaum's study (1978) proved to be subject to universal quantitative properties [5]. When the behavior of the system changes with this parameter, progressive dynamic transitions are observed, where in the range $0 < r < 1$, a pattern of extinction prevails as all iterations tend towards zero which represents the stable fixed point. As it passes $r = 1$ and enters the range $1 < r < 3$, the zero fixed point loses its stability, giving rise to a non-zero fixed point that represents the stable state of the system. When the critical value $r = 3$ is reached, the value of the derivative $\left| f_r'(x^*) \right|$ reaches -1, which leads to loss of stability and the appearance of bifurcation and chaotic behavior, and Figure (2) shows this mechanism geometrically, where the second iterate of the function appears at the beginning of the bifurcation $f_r^{(2)}(x)$. As the parameter r continues to increase in the range $3 < r < r_\infty \approx 3.5699...$, the system enters into a succession of period-doubling bifurcations, and Feigenbaum discovered that this sequence converges towards an accumulation point r_∞ ,where the ratios between the intervals at which the period doubling occurs decrease in a constant proportion known as Feigenbaum's constant δ, given by the following universal relation [5]:

$$\delta = \lim_{n \to \infty} \left(\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \right) \approx 4.6692016091029\ldots \tag{6}$$

Chaotic behavior becomes predominant in the parameter range $r_\infty < r \leq 4$, with the appearance of narrow periodic windows embedded in chaos, in which the system temporarily reverts to stable periodic behavior. It should be noted that the characteristic universal constants of these transitions, specifically the scaling constant α, known as Feigenbaum's second constant and given by the relation [5]:

$$\alpha = \lim_{n \to \infty} \frac{d_n}{d_{n+1}} \approx 2.5029078750957\ldots \tag{7}$$

This constant appears in a wide class of one-dimensional maps possessing a quadratic maximum, independent of the specific functional form of the map, reflecting the universality of this phenomenon [5].

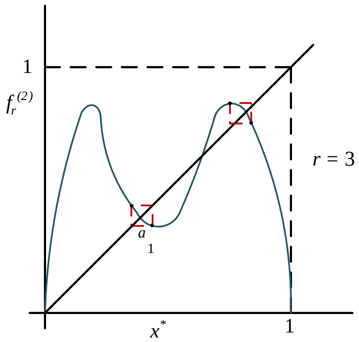

Figure 1: Schematic representation of the second iterate at the onset of bifurcation ($f_r^{(2)}(x)$ at $r = 3$).

The figure shows the transition from a fixed point to a period-2 cycle. The marked squares highlight the self-similar structure and the scaling factor α that characterizes the renormalization process (inspired by [5]).

The numbers α and δ are not determined by the set of derivatives of *f* at a given point, whether *f* is analytic or not, but rather there are universal functions that describe the local structure of stability sets, and these functions obey functional equations independent of *f* that implicate α and δ in a fundamental way [5].

2.1.2 The mathematical derivation of the Lyapunov exponent in one-dimensional maps

The study of the evolution of perturbations in tangent space quantitatively explains the concept of deterministic chaos. Assuming that there are two trajectories, one primary starting from the point x_0, and the other a nearby trajectory starting from the point $x_0 + \delta_0$, where $\delta_0 \to 0$, after n iterations applied to the first step [2]:

$$f(x_0 + \delta_0) \approx f(x_0) + f'(x_0) \cdot \delta_0 \Rightarrow \delta_1 \approx f'(x_0) \cdot \delta_0 \tag{8}$$

When the chain rule is applied to n iterations, the value of the perturbation δ_0 becomes the product of the derivatives:

$$\delta_n \approx \left\lfloor \prod_{i=0}^{n-1} f'(x_i) \right\rfloor \cdot \delta_0 \tag{9}$$

where i is the iteration index.

It is clear from Equation 9 that when there is a large number of iterations (which will be explained later), a very large number of mathematical operations will be performed and thousands of numbers will be multiplied, and this may lead to numerical overflow or underflow. The mathematical problem has been avoided by using the natural logarithm ln, in order to take advantage of the properties of the logarithm that convert the product into a sum [6], thus Equation 9 reads as follows:

$$\ln|\delta_n| \approx \sum_{i=0}^{n-1} \ln|f'(x_i)| + \ln|\delta_0| \tag{10}$$

By dividing both sides by the iteration step n and taking the limit as $n \to \infty$, the dependence on the initial perturbation δ_0 is eliminated, and the precise mathematical formula defining the Lyapunov exponent is obtained, given by the following relation [1]:

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \tag{11}$$

The time average can be replaced by the spatial average, based on the Oseledets Multiplicative Ergodic Theorem and Birkhoff's Ergodic Theorem, where it is represented using integration with respect to the ergodic invariant measure μ [4], so that equation 11 becomes as follows:

$$\lambda = \int \ln|f'(x)| d\mu(x) \tag{12}$$

In this context, the system's trajectory implies that, in the long run, the orbit explores the attractor ergodically, making the time average along a single trajectory exactly equivalent to the spatial average over the entire attractor, which justifies relying on one long trajectory to calculate λ rather than averaging over a large ensemble of initial conditions [4].

It is clear from Equation 12 that the approximate value of the Lyapunov exponent λ does not depend on the randomly selected initial point if and only if the system x_0 moves on an ergodic set [6]. When substituting Equation 3 in Equation 10, the final mathematical formula for the numerical calculation of the Lyapunov exponent is obtained in the logistic map with the following relation:

$$\lambda(r) \approx \frac{1}{N}\sum_{i=0}^{N-1}\ln\left|r\cdot(1-2x_i)\right| \tag{13}$$

The scientists Benettin [2] and Wolf [7] have temporarily abandoned the one-dimensional assumption in favor of Lyapunov's conceptual and perceptual clarification, since dynamic systems have been assumed to operate in a phase space of n-dimensions. Assuming that the three-dimensional phase space (value of $N = 3$), where a very small dot sphere (infinitesimal sphere) is defined consists of initial conditions surrounding a fiducial trajectory in the phase space. As time *t* (or the number of iterations n progresses), nonlinear dynamics causes the sphere to stretch in specific directions and compress in other directions, followed by folding the sphere to return it to its original size. Consecutive and unequal contraction and expansion affect the sphere to deform it into an n-ellipsoid [4].

2.1.3 Oseledets' theorem and tangent space deformation

Oseledets Multiplicative Ergodic Theorem states that the exponential growth rate as $n \to \infty$ of the axes of this ellipsoid converges towards fixed values called the Lyapunov spectrum [1]:

$$\lambda_k = \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{p_k(n)}{p_k(0)}\right), \quad k = 1, 2, \dots, N \tag{14}$$

where $p_k(n)$ is the length of the *k*-$^{\text{th}}$ axis of the ellipsoid at iteration *n*, and the exponents are ordered in descending order $\lambda_1 \geq \ \lambda_k \geq \cdots \geq \lambda_N$.

The previous definition explains that the maximum exponent λ_1 represents the maximum possible expansion of the system, and is directly responsible for the sensitivity of the system to the initial conditions as well as the loss of predictability. In addition, the sum of any set of exponents represents a certain rate of change in volume, and a system in which the sum of the exponents is negative is called a dissipative system ($\sum \lambda_i < 0$) [1, 2], which means that the total size of the phase space shrinks with time despite the expansion in certain directions. It is the definition of the basic geometric condition for the formation of a strange attractor [7].

When applied to a one-dimensional logistic map ($N = 1$), there is neither a sphere nor an ellipsoid. Expansion and contraction occur on a single axis expressed as a line segment. This leads to the reduction of the Jacobian matrix to become a mere scalar derivative. The Lyapunov spectrum is limited to one value only λ. It follows from the above that following a single axis is quite sufficient to describe the dynamics of bifurcation and chaos in this one-dimensional model [2, 7].

2.1.4 Dynamic and informational interpretation of λ values

A physical explanation and dynamic semantics expressing the numerical values of the Lyapunov exponent λ can be provided based on the following equation [6]:

$$e^{\lambda n} \approx \frac{|\delta_n|}{|\delta_0|} \tag{15}$$

When $\lambda = 0$, the separation between trajectories is exponentially reduced ($e^{\lambda n} \to 0$), which means that any small perturbation will disappear with time, and the convergent paths will converge towards a periodic attractor or fixed point. In this case, the system is considered to experience information loss regarding its initial conditions over time, making its long-term behavior stable and predictable.

When perturbations neither grow ($\lambda > 0$) nor fade exponentially ($\lambda < 0$), this corresponds to ($\lambda = 0$) occurring mathematically at the exact bifurcation points. In the logistic map at $r = 3$, the derivative at the fixed point is $f_r'(x^*) = -1$, and $\ln|-1| = 0$, this indicates that the system reaches a critical state at which it transitions from one dynamic pattern to another, resulting in a loss of linear stability.

When $\lambda > 0$, this represents the most important condition, because the disorder grows exponentially over time. The Lyapunov exponent is related to Kolmogorov-Sinai entropy in this case. The Lyapunov exponent λ is seen as the rate at which information is produced in chaotic systems, quantified by the positive value of λ (bits/iteration) [4], where the amount of new information produced by the system increases with its increase and thus increases the chaos, which leads to a loss of predictability, as well as a decrease in the predictability horizon.

This previous concept is expressed as a physical quantity called Lyapunov time or predictability horizon, which is given by relation [6]:

$$T_{\text{pr}} \approx \frac{1}{\lambda}\ln\left(\frac{\varepsilon}{\delta}\right) \tag{16}$$

where:

- δ: Initial measurement uncertainty,
- ε: is the maximum tolerable prediction error.

Equation 16 proves that the inverse proportionality between T_{pr} and λ means that extreme chaos (a large positive λ) undermines predictability too quickly

2.1.5 Finite-Time Lyapunov Exponents

In theoretical research, the Lyapunov exponent is defined as the limit as $n \to \infty$, but in numerical and programmatic application, as will be done in the third paragraph of this paper, an approximate value is calculated based on a finite number of iterations N (e.g., $N = 10^6$). The finite-time Lyapunov exponent is defined as $\Lambda(n)$ by the following equation [6]:

$$\Lambda(n) = \frac{1}{n}\sum_{i=0}^{n-1}\ln\left|f'(x_i)\right| \tag{17}$$

$\Lambda(n)$ represents a random variable (stochastic variable) that fluctuates around the true value λ. The distribution of these fluctuations follows the Large Deviation Theory.

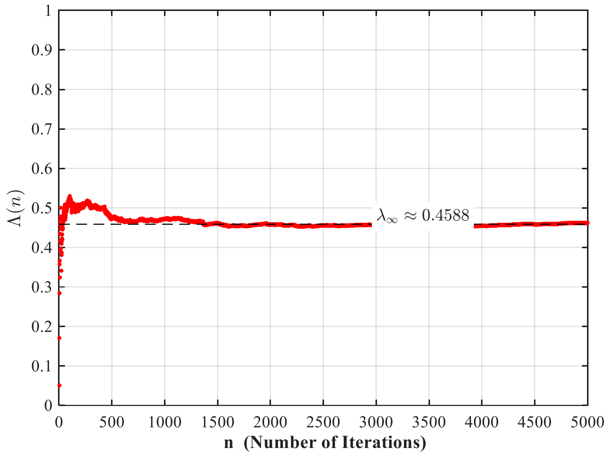

Figure 2: Convergence of the finite-time Lyapunov exponent $\Lambda(n)$ towards the asymptotic value $\lambda \approx 0.4588$ at a control parameter of $r \approx 3.88$ (inspired by [6]).

The graph illustrates the rapid decay of initial transients (transient phase) and the stability of the time average around a positive constant value, confirming the chaotic behaviour of the system and justifying the reliance on a finite number of N iterations in numerical calculations rather than the mathematical limit $n \to \infty$.

The practical effects of finite-time Lyapunov exponents are illustrated in this research as follows:
When plotting λ as a function of r, the random fluctuations of $\Lambda(n)$ will appear as slight noise in the curve, especially in chaotic regions with positive values, as it is a physical-mathematical property of chaotic systems that reflects the irregularity of the temporal evolution [6]. Inside each periodic window (e.g., at $r \approx 3.83$), where $\lambda < 0$, if the number of iterations N is small, $\Lambda(n)$ may take transient positive values before it settles, which may cause a spurious indication of chaos, which makes the selection of a relatively large value of N (e.g., 10^6) and the discarding of transients necessary to reduce these fluctuations [7]. Analyses based on Oseledets' theorem suggest

that fluctuations of $\Lambda(n)$ occur at asymmetric rates in tangent space, as contraction rates can be much faster in magnitude than expansion rates. This explains the existence of sharp downward oscillations (large negative values) in the transition zones between chaos and order [1].

2.2 Methodology

The aim of this research is to develop a robust numerical algorithm for estimating Lyapunov exponents for logistic maps, taking into account the computational constraints and numerical accuracy requirements. The proposed methodology is based on the concept of finite-time Lyapunov exponents, which is a fundamental approach in the analysis of nonlinear dynamical systems [6].

The proposed algorithm is based on the Lyapunov exponent definition for discrete one-dimensional maps, where the exponent is estimated via the time average of the natural logarithm of the absolute value of the map derivative according to Equation 13. The algorithm is based on a dynamical simulation of the system at each value of the control parameter r, and consists of four sequential and interrelated phases.

2.2.1 Initialization

A random initial condition is chosen $x \in [0,1]$, and based on the ergodic properties of the logistic map within the chaotic regime, the selection of any initial value in the open interval (0,1) will eventually lead to an accurate estimate of the Lyapunov exponent, after discarding the transient phase in the parameter range $r \in [3.57, 4]$ [6]. The central value $x_0 = 0.5$ is chosen in order to ensure symmetry and reduce potential numerical bias.

2.2.2 Discarding Transients

This step contributes to ensuring the accuracy of the final estimate, since when the simulation is started from the point x_0, the system spends a number of iterations N_t in a transient phase before its trajectory converges to the attractor [7]. Calculating the derivatives during this phase will produce values that do not reflect the properties of the true attractor but only the properties of the transient dynamics. A number of initial iterations N_t were performed and their results were completely excluded from the final arithmetic mean in order to allow the system to converge to the attractor. The standard number of discarded iterations is $N_t = 1000$, a value typically sufficient to ensure convergence to the attractor even in regions with complex dynamics.

2.2.3 Accumulation Phase

After confirming that the system has reached the attractor, additional N iterations are performed (in this research the number $N = 10^6$). Within this stage, the following repetitions are performed in each iteration i:

1. State update: Calculate the following point according to the logistic map from Equation 2:
$$x_{i+1} = r \cdot x_i (1 - x_i)$$
2. Derivative calculation: the evaluation of the derivative at the current point of Equation 3:
$$f'(x_i) = r \cdot (1 - 2x_i)$$
3. Logarithmic transformation: calculating the contribution to the Lyapunov exponent from the following equations:
$$l_i = \ln \left| r \cdot (1 - 2x_i) \right|$$
4. Accumulation: Adding the value l_i to the cumulative sum S:
$$\mathrm{S} = \mathrm{S} + l_i$$

2.2.3 Final Estimation

After completing N iterations in the accumulation phase, the Lyapunov exponent for the parameter value r is estimated by Equation 17, and the finite-time estimate accounts for the excluded transients as follows:

$$\lambda(r) = \frac{1}{N} \sum_{i=N_t+1}^{N_t+N} \ln \left| r \cdot (1 - 2x_i) \right|$$

As explained in Section 2.1.3, for the one-dimensional logistic map in this paper, the use of algorithms that rely on the tracking of orthonormal vector frames in tangent space and the use of Gram-Schmidt orthonormalization [2, 7] is unnecessary in this research, because the tangent space is simply a line, and the Jacobian reduces to a scalar derivative; in addition, the use of logarithms to convert products into sums [6] ensures numerical stability. This is what makes the proposed algorithm the most accurate and computationally efficient for this particular model.

2.2.4 Parameter Scan and Mapping the Full Spectrum

This paper presents the complete Lyapunov exponent spectrum, which is a graph showing the variation of λ as a continuous function of the control parameter r. To achieve this, the above algorithm is applied within a loop that comprehensively scans the values of r:

Range Specification: The range was chosen as $r \in [2.5, 4]$, because values below the minimum quickly lead the system towards stability where λ values are very negative and relatively constant, as this domain $r < 2.5$ does not add analytical value to this study. At the upper limit, the logistic map loses the invariance of the interval [0,1], as for $r > 4$ trajectories may diverge to $-\infty$ for almost any initial condition $x \in [0,1]$, which makes the analysis cease to describe bounded dynamics in this context.

Determining the Scan Resolution: A high parameter resolution has been adopted in order to capture the fine details of the dynamics, especially the very narrow periodic windows embedded in the chaotic regime (e.g., the period-3 window near $r \approx 3.83$ [5]). A step size of $\Delta r = 0.001$ was chosen. This means that the final graph will consist of 1500 data points, which provides a balanced trade-off between visual accuracy and computational processing time.

2.2.5 Pseudocode of the proposed algorithm

Table 1 The proposed algorithm is illustrated using a pseudocode language

ALGORITHM 1: LYAPUNOV EXPONENT FOR LOGISTIC MAP

```
Input: $r_{min}$, $r_{max}$, $\Delta r$, $N_t$, $N$
for $r \leftarrow r_{min}$ to $r_{max}$ step $\Delta r$ do
    $x \leftarrow 0.5$
    for $i \leftarrow 1$ to $N_t$ do $x \leftarrow r \cdot x \cdot (1 - x)$
        $S \leftarrow 0$
        for $j \leftarrow 1$ to $N$ do
            $x \leftarrow r \cdot x \cdot (1 - x)$
        end for
    end for
    $\lambda(r) \leftarrow S/N$
    store $\lambda(r)$
end for
Output: $\lambda(r)$ array
```

This computational structure ensures that the data to be generated will be accurate enough to compare with the bifurcation diagram and reveal the smallest dynamical details of the system.

The proposed algorithm was implemented using vectorization in the MATLAB environment, rather than relying on traditional nested loops. This technique relies on performing calculations on complete arrays in a single operation, which significantly reduces the computational burden of nested loops and takes advantage of MATLAB's optimized mathematical libraries.

To assess the impact of this technique, the execution time between the two methods was compared. Using traditional loops, the execution time was about 335 seconds at a control parameter array r of length 10000. Using the vectorization technique, the time was reduced to only 0.34 seconds, an improvement of more than 1000 times. This improvement allowed the number of iterations N in the accumulation phase to be increased to one million (10^6) while still achieving an acceptable execution time of only 2.38 seconds. Allowing the use of a large number of iterations ensures:

1. Higher accuracy in calculating the Lyapunov exponent λ.
2. Better convergence of results.
3. Revealing fine-scale dynamical details of the system.

3 Results and discussion

3.1 Results and analysis

3.1.1 Variation of the Lyapunov exponent λ as a function of the control parameter r

Numerical simulations for the computation of the Lyapunov exponent λ over the parameter range $r \in [2.5, 4]$ revealed a characteristic behavior that reflects the dynamical regime of the logistic map, as shown in Figure 3. The curve exhibits a transition from negative values ($\lambda < 0$), indicating stable periodic behavior, to positive values ($\lambda > 0$), indicating deterministic chaos. In the chaotic regime ($r < r_\infty \approx 3.5699\ldots$), the spectrum shows sharp variations and frequent negative excursions. These intervals where $\lambda < 0$ correspond to periodic windows embedded in the chaotic regime, during which the system exhibits stable periodic behavior and temporarily behaves in a regular manner amid the chaos.

3.1.2 Quantitative correlation with bifurcation diagram

The bifurcation diagram in Figure 4 and the Lyapunov spectrum shown in Figure 3 were compared in order to verify the accuracy of the quantitative index provided by the Lyapunov exponent λ, showing a qualitative and quantitative agreement that is summarized as follows:

Stable fixed-point regime ($2.5 < r < 3$): The orbits in the bifurcation diagram converge to a single value of x, and $\lambda < 0$ in this range confirms the stability of the fixed point and the exponential decay of any initial perturbation with time.

Period-doubling cascade ($r < r_\infty \approx 3.5699\ldots$): As r increases, the stable fixed point undergoes a sequence of period-doubling bifurcations into period-2, period-4, period-8, ... orbits, and the Lyapunov exponent λ gradually approaches zero from below, touching it exactly at bifurcation points (e.g., $r = 3$), where the solutions lose their linear stability ($\left| f_r'(x^*) \right| = 1$). The exponent λ remains negative along this sequence, confirming that the system is still periodic and stable despite its increasing complexity, and has not yet entered a state of deterministic chaos.

Chaotic regime ($r_\infty < r \leq 4$): Beyond the accumulation point r_∞, the bifurcation diagram exhibits a fractal structure characteristic of aperiodic behavior, and positive values of λ predominate in this range. The quantifiable positive value illustrates the phenomenon of sensitive dependence on initial conditions, which is the defining characteristic of deterministic chaos.

Periodic windows embedded in the chaotic regime: The period-3 window near $r \approx 3.83$ is a well-documented quantitative phenomenon, in which the apparently chaotic behavior in the bifurcation diagram collapses into three distinct branches (period-3 orbit), in which λ drops sharply below the zero axis towards negative values. This demonstrates the sensitivity and accuracy of the Lyapunov exponent in detecting stable periodic orbits embedded within chaos, a task that is beyond the scope of visual analysis of the bifurcation diagram alone.

3.1.3 Informative Significance and Measurement of Chaos Severity

The Lyapunov exponent λ is a continuous measure of the rate of exponential divergence or the rate of loss of predictability, as shown in Figure 3, where λ varies with the control parameter r within the chaotic regime and reaches its theoretical maximum value $\lambda_{max} = \ln 2 \approx 0.693$ at $r = 4.0$. According to the theoretical framework of chaos [4], the positive value of λ is directly related to the Kolmogorov-Sinai entropy (the rate of information production in bits/iteration), and therefore the higher the value of λ, the greater the rate of exponential divergence of nearby trajectories, and the shorter the prediction horizon (Lyapunov time) of the system. This confirms that the exponent detects chaos and quantifies both the dynamical instability and the rate of information production with quantitative accuracy.

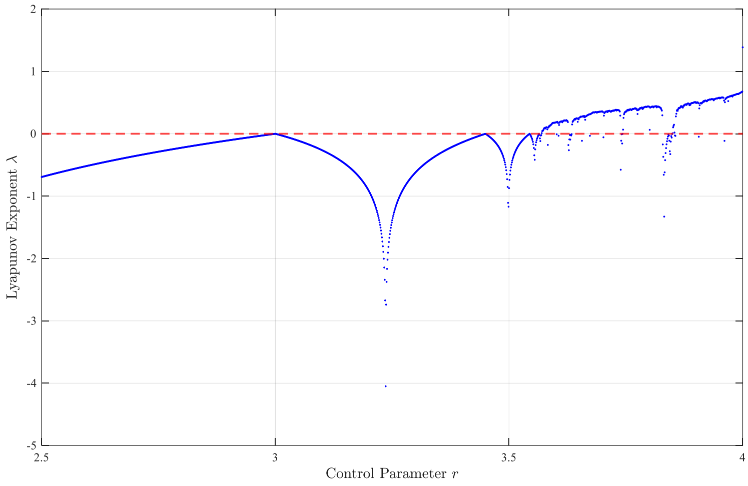

Figure 3: Spectrum of Lyapunov exponent (λ) as a function of the control parameter r for the logistic map.

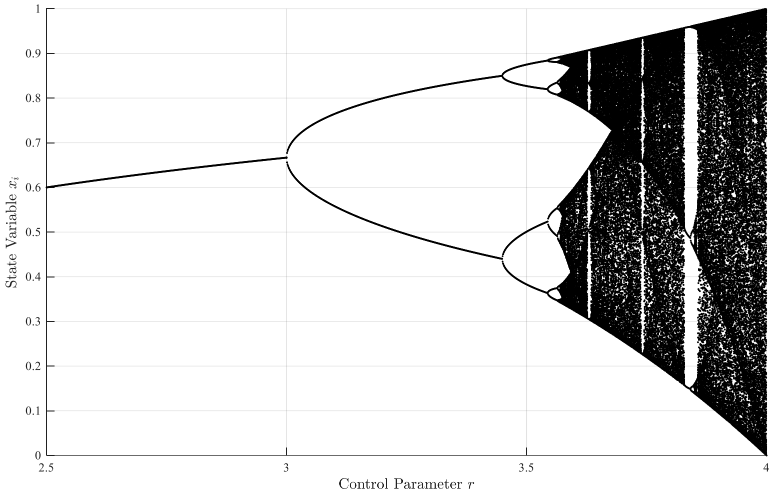

Figure 4: A bifurcation diagram showing the transition from stability to chaos through the period-doubling cascade of the logistic map

3.2 Discussion

3.2.1 Lyapunov exponent as a quantitative detection tool for chaos

The systematic comparison between the bifurcation diagram and the Lyapunov spectrum, as described this Section, shows a functional distinction for quantitative characterization based on Lyapunov foundations compared to visual inspection, in which the bifurcation diagram provides an intuitive representation of the structure and branches of phase space, but the visual distinction between high-order periodicity and weak chaotic behavior remains subject to ambiguity, especially in regions of complex dynamical structure. The results obtained emphasize that the Lyapunov exponent λ objectively resolves this confusion, as the sharp drop to negative values at $r \approx 3.83$ reveals the presence of a stable period-3 window, a phenomenon that may be mistakenly interpreted as chaotic behavior if visual analysis is based on inspection alone.

This revealing behavior is consistent with the theoretical framework established by Eckmann and Ruelle [4], where a chaotic system is mathematically defined by the presence of at least one positive Lyapunov exponent, which reflects the property of sensitive dependence on initial conditions. From the above, it can be seen that the Lyapunov exponent not only locates the system in parameter space, but also quantifies the nature of its dynamics through a continuous numerical scale, providing an objective criterion for distinguishing chaotic from periodic regimes independent of the observer's visual subjectivity.

3.2.2 Dependencies and Applications

The significance of the Lyapunov exponent λ goes beyond merely the qualitative classification of dynamics, providing deep quantitative insights into the informational and physical capacity of a system. As Pikovsky and Politi [6] explained, the positive Lyapunov exponent is directly related to the Kolmogorov-Sinai entropy (metric entropy), which expresses the rate at which the system produces new information per iteration. The results in this research suggest that the value of λ increases as r approaches 4.0, reflecting an acceleration in the rate of information generation, resulting in a rapid erasure of the memory of the initial conditions.

The prediction horizon (Lyapunov time) T_{pr} is inversely proportional to the value of λ, with the results showing that at $r \approx 4.0$, the prediction horizon becomes too short, making long-term prediction practically impossible. This principle applies broadly across multiple disciplines, from modeling weather systems (where Lorenz first discovered these dynamics), to population dynamics, to chaos-based cryptography applications, where a high exponent ensures sufficient complexity to deter attackers from restoring or anticipating encrypted signals.

3.2.3 Limitations of the study

Although the numerical algorithm used (based on Benettin's [2] approach to one-dimensional maps) is effective, it is subject to fundamental limitations associated with numerical calculations. The calculation in this paper is based on a finite number of iterations ($N = 10^6$), which means that the Lyapunov exponent is estimated as a finite-time Lyapunov exponent rather than the theoretical limit as $n \to \infty$. As Pikovsky and Politi explain [6], finite-time Lyapunov exponents represent random variables that fluctuate around the true asymptotic value, which explains why there is a slight noise in the Lyapunov spectrum, especially in chaotic regions where the derivative $f'(x)$ changes dramatically.

The proposed algorithm relied on a single trajectory starting from the initial condition $x_0 = 0.5$. This approach is justified by ergodic theory, which states that for the logistic map in a chaotic regime, the time average along a single trajectory is equivalent to the spatial average over the attractor [4]. However, in systems with multiple basins of attraction, the choice of x_0 may introduce a significant bias to the results. Wolf et al. [7] point out that empirical data often require more complex techniques for phase space reconstruction (such as delay-coordinate embedding combined with Gram-Schmidt reorthonormalization), while this research has benefited from explicit knowledge of the map's functional form, allowing for a direct and accurate calculation of the derivative.

Conclusion

This study proved successful in using the Lyapunov exponent λ as a quantitative criterion for detecting and characterizing chaos within the logistic map, where a clear and objective picture of the dynamical transformations of the system was presented by applying a numerical algorithm to calculate the finite-time Lyapunov exponent spectrum within the control parameter range of $2.5 \leq r \leq 4.0$.

The results showed a strong correspondence between the Lyapunov spectrum and the bifurcation diagram, with negative exponent values accurately correlated with stable fixed points and periodic orbits, while positive values clearly indicated the onset of chaotic behavior. The sensitivity of the Lyapunov exponent was also evident in its ability to detect narrow periodic windows embedded in chaos, most notably the period-3 window near $r \approx 3.83$, where the system temporarily returns to regular behavior within the chaotic regime.

This paper reaffirms the theoretical predictions of nonlinear dynamics, as the Lyapunov exponent serves not merely as a binary indicator of the presence of chaos, but as a fundamental measure of system stability and its rate of information production. This method bridges the gap between qualitative geometric observations and rigorous quantitative analysis, demonstrating that it is an indispensable tool for understanding complex dynamical behaviors, both in mathematical models and in real-world physical systems.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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