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An expands Definition the Random Variable in Probability Theory

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توسيع تعريف المتغير العشوائي في نظرية الاحتمالات

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Abstract:
This paper clarifies how to deal with the random variable expands its definition, and recognizes the statistical measures of random processes, such as prediction and variance in a more appropriate form for random processes, to work on the random variable and link these processes to real processes through a real function "The density function". It also focuses on discussing the characteristics of the statistical independence of the random variable and the ability to work on it in terms of new definitions. Then clarify the idea of a random convergence to a random variable using "BOREL-CONTELLI LEMMA" to work on the stochastic differential equations.

Keywords: Random Processes, BOREL-CONTELLI LEMMA, Density Function, Stochastic Differential Equation.

الملخص
توضح هذه الورقة البحثية كيفية التعامل مع المتغير العشوائي وتوسيع تعريفه، وتتعرف على المقاييس الإحصائية للعمليات العشوائية، مثل التوقع والتباين، في شكل أكثر ملائمة للعمليات العشوائية، للعمل على المتغير العشوائي وربط هذه العمليات بالعمليات الحقيقية من خلال دالة حقيقية " دالة الكثافة". كما تركز على مناقشة خصائص الاستقلال الاحصائي للمتغير العشوائي والقدرة على العمل عليه من حيث تعريفات جديدة. ثم توضيح فكرة التقارب العشوائي لمتغير عشوائي باستخدام " BOREL-CONTELLI LEMMA " للعمل على المعادلات التفاضلية العشوائية.

الكلمات المفتاحية: العمليات العشوائية، مبرهنة بوريل كونتيلي، دالة الكثافة، المعادلة التفاضلية العشوائية.

Introduction

Most of differential equations that were developed to describe physical phenomena have ignored stochastic effects due to the difficulty of dealing with it as a real variable. This paper measure's theoretic introduction to stochastic process, and as such assumes a knowledge of calculus and elementary probability process. In which we attempt to present some of expanded definitions, lemmas and theory of converges stochastic process, like previous studies [4-7,10,12,13], to clarify its diverse range of applications same as [11,14], and the function that describes the physical system can be a random function. This random function is suitable for describing the rapidly fluctuating random phenomena and can be worked on by a Brownian motion or Wiener process see [1-3, 9] so this paper helps everyone who cares about stochastic analysis and stochastic differential equation.

Probability Space

We must carefully define what we mean by the term "random". The correct way to do so is by introducing the precise mathematical structure of a probability space. We start with a nonempty set, denoted by Ω, certain subsets of which we will interpret as being "events" in a moment. There are two natural equations that one might ask:

1) How can we construct a probability space on which a stochastic process is defined?
2) Is it possible to define a stochastic process by specifying, say, its finite dimensional distributions?

Definition 1.1. [4-5] A σ-algebra is a collection F of subsets of Ω with these properties:

i. $\emptyset, \Omega \in F$.
ii. If $A \in F$, then $A^c \in F$.
iii. If $A_1, A_2, \cdots \in F$, then $\bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in F$. Here $A^c := \Omega - A$ is the complement of A.

Definition 1.2. [4-5] A triple (Ω, F, P) is called a probability space provided Ω is any set, F is a σ-algebra of subsets of Ω, and P is a probability measure on Ω.

Example 1.3. [8]The smallest σ-algebra containing all the open subsets of $\mathbb{R}^n$ is called the Borel σ-algebra, denoted $\mathfrak{B}$. Assume that f is a nonnegative, integrable function, such as $\int_{\mathbb{R}^n} f(x)\,dx = 1$. We define $P(B) := \int_B f(x)\,dx$ for each $B \in \mathfrak{B}$.Then $(\mathbb{R}^n, \mathfrak{B}, P)$ is a probability space. Call f the density of a probability measure P.

Definition 1.4. [4] Let (Ω, F, P) be a probability space. And a mapping $X: \Omega \to \mathbb{R}^n$ is called an n-dimensional random variable if for each $B \in \mathfrak{B}$, we have $X^{-1}(B) \in F$. We equivalently say that X is F-measurable.

Notation:

1. Write X but not $X(\omega)$. This follows the custom within probability theory of mostly not displaying the dependence of random variables on the sample point $\omega \in \Omega$.
2. Will use without further comment various standard facts from measure theory, for instance that sums and products of random variables are random variables.

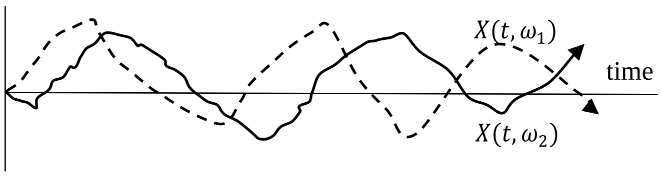
Lemma 1.5. Let $\mathrm{X}: \Omega \to \mathbb{R}^n$ be a random variable. Then $\mathrm{F(X)} := \{\mathrm{X}^{-1}(\mathrm{B}) \backslash \mathrm{B} \in \mathfrak{B}\}$ is a σ-algebra, called the σ-algebra generated by X. This is the smallest sub- σ-algebra of F with respect to which X is measurable.

2. Random Processes

We have random variables depending upon time.

Definition 2.1. [12,7] A collection $\{X(t) \backslash t \geq 0\}$ of random variables is called a stochastic process. For each point $\omega \in \Omega$, the mapping $t \mapsto X(t, \omega)$ is the corresponding sample path.

This mean that if we run an experiment and observe the random values of $X(\cdot)$ as time evolves, we are in fact looking at a sample path $\{X(t, \omega) \backslash t \geq 0\}$ for some fixed $\omega \in \Omega$. If rerun the experiment, get general observe a different sample path.



Figurer 1: Two sample paths of a stochastic process.

Definition 2.2. [12,7], If (Ω, F, P) is a probability space and $X = \sum_{i=1}^{k} a_i \mathcal{X}_{A_i}$ is a real-valued simple random variable, define the integral of X by $\int_{\Omega} XdP := \sum_{i=1}^{k} a_i P(A_i)$.

If X is nonnegative random variable, define $\int_{\Omega} XdP := \sup\limits_{Y \leq X, Y simple} \int_{\Omega} YdP$.

Now, if $X: \Omega \to \mathbb{R}^n$ is a random variable, then $\int_{\Omega} XdP := \int_{\Omega} X^+ dP - \int_{\Omega} X^- dP$, provided at least one of the integrals on the right is finite. And $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$, so that $X = X^+ - X^-$. And, we suppose $X: \Omega \to \mathbb{R}^n$ is a vector- valued random variable, $X = (X^1, X^2, \cdots, X^n)$. Then write $\int_{\Omega} XdP = \left(\int_{\Omega} X^1 dP, \int_{\Omega} X^2 dP, \cdots, \int_{\Omega} X^n dP\right)$.

Definition 2.3. [12] $E(X) \coloneqq \int_{\Omega} X dP$, is call the expected value (or mean value) of X. And $V(X) \coloneqq \int_{\Omega} |X - E(X)|^2 dP$, the variance of X.

Lemma 2.4. (Chebyshev's inequality). [12,7], If X is a random variable and $1 \le p < \infty$, then $P(|X| \ge \lambda) \le \frac{1}{\lambda^p} E(|X|^p)\ for\ all\ \lambda > 0$.

Definition 2.5. [5] The distribution function of X is the function $F_X: \mathbb{R}^n \to [0,1]$ defined by $F_X(x) \coloneqq P(X \le x)\ \ for\ all\ x \in \mathbb{R}^n$and if $X_1, \cdots, X_m: \Omega \to \mathbb{R}^n$are random variables, their joint distribution function is $F_{X_1,\cdots,X_m}: (\mathbb{R}^n)^m \to [0,1]$, $F_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_n} f(y_1, \cdots y_n) dy_n \cdots dy_1$, then $f(x)$ is called the density function for X. It follows that

$$P(X \in B) = \int_B f(x) dx \quad for\ all\ B \in \mathfrak{B} \qquad (1)$$

This formula is important as the expression on the righthand side is an ordinary integral, and can often be explicitly calculated.

Example 2.6. If $X: \Omega \to \mathbb{R}$ has density $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|x-m|^2}{2\sigma^2}}$ $(x \in \mathbb{R})$, we say X has a Gaussian (or normal) distribution, with mean m and variance σ^2. In this case let write X is an $N(m, \sigma^2)$ random variable.

3. CONVERGES IN PROBABILITY

Definition 3.1. [6,11] A continuous time stochastic process $X = \{X(t) \backslash t \ge 0\}$ is said to be stochastically continuous (or continuous in probability) for every $t \ge 0$, and for every $\varepsilon > 0$, $\lim_{s \to t} P(|X_t - X_s| > \varepsilon) = 0$, for $0 \le s < t < \infty$.

Lemma 3.2. [8] Let $X: \Omega \to \mathbb{R}^n$ be a random variable, and assume that its distribution function $F = F_X$ has the density f. Suppose $g: \mathbb{R}^n \to \mathbb{R}$, and $Y = g(X)$ is integrable. Then $E(Y) = \int_{\mathbb{R}^n} g(x) f(x) dx$. In particular $E(X) = \int_{\mathbb{R}^n} x f(x) dx$ and $V(X) = \int_{\mathbb{R}^n} |x - E(X)|^2 f(x) dx$.

Definition 3.3. [6,11] Tow events A and B are called independent if $P(A \cap B) = P(A)P(B)$, where $P(A) = P(A/B) = \frac{P(A \cap B)}{P(B)}$.

Theorem 3.4. [8] The random variables $X_1, \cdots, X_m: \Omega \to \mathbb{R}^n$ are independent if and only if $F_{X_1,\cdots,X_m}(x_1, \cdots, x_m) = F_{X_1}(x_1) \cdots F_{X_m}(x_m)$ for all $x_i \in \mathbb{R}^n, i = 1, \cdots, m$. If the random variables have densities, equivalent to $f_{X_1,\cdots,X_m}(x_1, \cdots, x_m) = f_{X_1}(x_1) \cdots f_{X_m}(x_m)$ for all $x_i \in \mathbb{R}^n, i = 1, \cdots, m$, where the function f are the appropriate densities.

Lemma 3.5. (BOREL-CONTELLI) [8] if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n i.o.) = 0$. Where $A_n i.o. = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \{\omega \in \Omega | \omega$ belongs to infinity many of the A_n $\}$.

Now by Borrel-contelli lemma a sequence of random variables $\{X_k\}_{k=1}^{\infty}$ defined on some probability space converges in probability to a random variable X, provided $\lim_{k \to \infty} P(|X_k - X| > \varepsilon) = 0$ for each $\varepsilon > 0$.

Theorem 3.6. [11] If $X_k \to X$ in probability, then there exists a subsequence $\left\{X_{k_j}\right\}_{j=1}^{\infty} \subset \{X_k\}_{k=1}^{\infty}$ such that $X_{k_j}(\omega) \to X(\omega)$ for every ω.

Proof: Let a positive integer j we select k_j large that $P\left(\left|X_{k_j} - X\right| > \frac{1}{j}\right) \le \frac{1}{j^2}$, and $\cdots < k_{j-1} < k_j < \cdots$, $k_j \to \infty$. Let $A_j \coloneqq \left\{\left|X_{k_j} - X\right| > \frac{1}{j}\right\}$. And since $\sum \frac{1}{j^2} < \infty$, then $P(A_n i.o.) = 0$. So, for all sample points ω, $\left|X_{k_j}(\omega) - X(\omega)\right| \le \frac{1}{j}$ provided $j \ge J$, for some index J depending on ω.

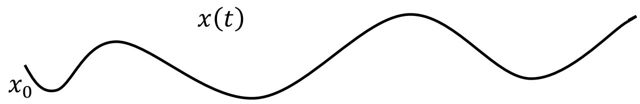
4. How to Harness This Technique to The Stochastic Differential Equation

We give some heuristics to stochastic differential equations and example of it.

Definition 4.1. [12] In any ordinary differential equation

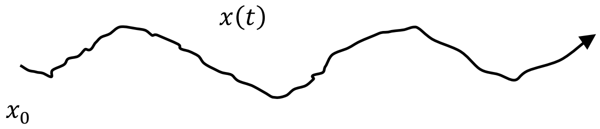
$$\begin{cases} \dot{x}(t) = b(x(t)) & t > 0 \\ \quad x(0) = x_0, \end{cases} \qquad (2)$$

where b: $\mathbb{R}^2 \to \mathbb{R}^2$ is given, smooth vector and the solution is the trajectory $x(\cdot)$: $[0, \infty) \to \mathbb{R}^2$.



Figurer 2: Two sample paths of a stochastic process.

In many applications, however, the experimentally measured trajectories of systems by (ODE) don't in fact behave as predicted.



Figurer 3: Two sample paths of a stochastic process.

This unexpected act in figurer 3 is called a white noise symbol $\xi(\cdot)$.

Definition 4.3. [3,10] A stochastic process $W(\cdot)$ is called a Brownian motion or Wiener process if

i-$W(0) = 0$ a.s.,

ii- $W(\mathrm{t}) - W(s)$ is $N(0, t - s)$ for all $t \geq s \geq 0$, where $N(0, t - s)$ the probability density at time $t - s$.

iii- for all times $0 < t_1 < t_2 < \cdots < t_n$, the random variables $W(t_1), W(t_2) - W(t_1). \cdots, W(t_n) - W(t_{n-1})$ are independent ("independent increments").

4.3. Stochastic differential equations.[1] Somehow to include the possibility of random effects disturbing the system. A formal way to do so is to write:

$$\begin{cases} \dot{X}(t) = b\big(X(t)\big) + B\big(X(t)\big)\xi(t) \quad (t > 0) \\ X(0) = x_0, \end{cases}$$

Where $B: \mathbb{R}^2 \to \mathbb{M}^{n \times m}$ (space of $n \times m$ matrices) and $\xi(\cdot) :=$ m-dimensional " white noise". When we write in the different form and use $dW(t) = \xi(t)\, dt$, where $dW(t)$ denotes differential from the Bronian motion, we obtain the stochastic differential equations

$$\begin{cases} \dot{X}(t) = b\big(X(t)\big) + B\big(X(t)\big)W(t) \quad (t > 0) \\ X(0) = x_0, \end{cases}.$$

Conclusion

In conclusion, this paper aims to provide an in-depth exploration of stochastic analysis, specifically focusing on the application of probability theory and stochastic calculus. The main objectives include demonstrating processes involving the random element of random analysis, analyzing it through expected variance, and establishing connections with real analysis through the use of real functions. Additionally, the convergence of the random element for solving stochastic differential equations is addressed. By achieving these goals, this research contributes to advancing our understanding and utilization of stochastic analysis in various fields.

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