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Limit Cycles in a Reduced Adaptive Predator–Prey Model (AV-PP) with Nonlinear Functional Responses

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الدورات الحدّية في نموذج مفترس–فريسة تكيفي مُختزل مع استجابات وظيفية غير خطية

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Abstract:
This study investigates the emergence of limit cycles in adaptive predator–prey models by combining rigorous model reduction techniques with nonlinear functional responses. The analysis begins with a three-dimensional adaptive vulnerability predator–prey (AV-PP) model, in which prey susceptibility is treated as a dynamic variable to enhance biological realism. Due to the analytical complexity of the system, it is reduced to a two-dimensional planar model using a quasi-steady-state approximation (QSSA) supported by singular perturbation theory. This reduction is theoretically justified and preserves the essential qualitative dynamics of the original system. The reduced model with a linear interaction term is first examined, and the results show convergence toward a stable equilibrium without oscillatory behavior, indicating the absence of limit cycles. This highlights the limitation of linear formulations in representing realistic ecological dynamics. To address this, nonlinear functional responses are introduced. The incorporation of a Holling Type II functional response leads to equilibrium destabilization under specific parameter conditions, resulting in the emergence of a stable limit cycle. Further extension using a Holling Type III response enhances the robustness and persistence of oscillations by capturing ecological mechanisms such as predator learning and prey refuge effects. The qualitative analysis is carried out using the Poincaré–Bendixson theorem and the Elemental Limit Cycle (ELC) method. The results demonstrate that while the reduction simplifies the system without altering its fundamental behavior, the introduction of nonlinear functional responses is the key mechanism driving the transition from stable equilibrium to sustained oscillatory dynamics.

Keywords: Adaptive Predator–Prey Model, Limit Cycles, Functional Response, Elemental Limit Cycle (ELC) Method, Poincaré–Bendixson Theorem.

الملخص
تهدف هذه الدراسة إلى تحليل ظهور الدورات الحدّية في نماذج المفترس–الفريسة التكيفية من خلال الجمع بين تقنيات الاختزال الرياضي والاستجابات الوظيفية غير الخطية. يبدأ التحليل بنموذج ثلاثي الأبعاد يعتمد على القابلية التكيفية للفريسة، حيث تُعامل هذه القابلية كمتغير ديناميكي لتحقيق تمثيل بيولوجي أكثر واقعية. ونظرًا لتعقيد النموذج، يتم اختزاله إلى نظام

ثنائي الأبعاد باستخدام تقريب الحالة شبه المستقرة (QSSA) المدعوم بنظرية الاضطراب المفرد، وهو اختزال قائم على أساس نظري صارم ولا يؤثر على الخصائص النوعية الأساسية للنظام. تمت دراسة النموذج المختزل في حالته الخطية، حيث أظهرت النتائج تقارب الحلول نحو نقطة توازن مستقرة دون ظهور تذبذبات دورية، مما يؤكد عدم وجود دورات حدّية ويبرز محدودية النماذج الخطية في تمثيل السلوك البيئي الواقعي. ولتجاوز هذه القيود، تم إدخال استجابات وظيفية غير خطية. أدى إدماج دالة هولنق من النوع الثاني إلى زعزعة استقرار نقطة التوازن ضمن شروط معينة، مما أدى إلى ظهور دورة حدّية مستقرة. كما ساهم استخدام دالة هولنق من النوع الثالث في تعزيز قوة واستمرارية التذبذبات من خلال تمثيل سلوكيات بيولوجية مثل تعلم المفترس وتأثير الملاذات. تم إجراء التحليل باستخدام مبرهنة بوانكاريه–بندكسون وطريقة الدورات الحدّية الأولية(ELC) ، حيث أظهرت النتائج أن الاختزال يسهم في تبسيط النموذج دون التأثير على سلوكه الأساسي، في حين تمثل الاستجابات غير الخطية العامل الحاسم في انتقال النظام من حالة التوازن إلى السلوك التذبذبي المستمر.

الكلمات المفتاحية: نموذج مفترس–فريسة تكيفي، الدورات الحدّية، دوال الاستجابة، طريقة الدورات الحدّية الأولية(ELC) ، مبرهنة بوانكاريه–بندكسون.

Introduction

Predator–prey models constitute a fundamental framework for understanding population dynamics in ecological systems. Classical formulations, particularly the Lotka–Volterra model, provide a simple mathematical representation of species interactions; however, their reliance on linear interaction terms often limits their ability to capture realistic biological behavior ([5]; [11]). In particular, such models may produce neutral oscillations or overly simplified dynamics that are highly sensitive to initial conditions, making them inadequate for describing real ecological systems [7].

In natural ecosystems, predator–prey interactions are governed by complex nonlinear mechanisms, including predator saturation, handling time, and adaptive behavioral responses. These biological constraints cannot be adequately represented by linear models, which assume unlimited predation rates. To address these limitations, nonlinear functional responses have been introduced, providing a more realistic description of trophic interactions ([2]; [3]). The incorporation of such nonlinearities significantly alters the qualitative behavior of the system and may lead to the emergence of sustained oscillatory dynamics in the form of limit cycles [6].

From a mathematical perspective, the study of limit cycles plays a central role in nonlinear dynamical systems. These periodic solutions represent self-sustained oscillations that arise from the intrinsic structure of the system rather than external forcing [9]. Their existence can often be established using qualitative analytical tools such as the Poincaré–Bendixson theorem, which provides rigorous conditions for the presence of periodic orbits in planar systems ([8]; [4]). In addition, simplified analytical approaches such as the Elemental Limit Cycle (ELC) method have been developed to facilitate the study of oscillatory behavior in biological models without requiring complex transformations [10].

To improve biological realism, higher-dimensional predator–prey models have been proposed by incorporating additional ecological variables, such as adaptive prey vulnerability. While these models provide a more accurate representation of ecological interactions, their increased dimensionality introduces significant analytical challenges, particularly in the study of periodic solutions. Since key theoretical tools, including the Poincaré–Bendixson theorem, are restricted to two-dimensional systems, it becomes necessary to reduce such models to a planar form.

In this study, a three-dimensional adaptive vulnerability predator–prey model is considered and systematically reduced to a two-dimensional system using a quasi-steady-state approximation based on time-scale separation. This reduction is theoretically justified and preserves the essential qualitative dynamics of the original system, allowing for rigorous analysis of stability and oscillatory behavior.

The reduced model is first analyzed under a linear interaction assumption, where the system is shown to converge toward a stable equilibrium without exhibiting periodic oscillations. This result highlights the limitation of linear formulations in capturing realistic ecological dynamics. To overcome this limitation, the model is subsequently extended by incorporating nonlinear functional responses, specifically the Holling Type II and Type III forms.

The main objective of this study is to investigate how the introduction of nonlinear functional responses alters the dynamical behavior of the system, leading to the emergence of limit cycles. The analysis is conducted using the

Poincaré–Bendixson theorem in combination with the Elemental Limit Cycle (ELC) method, providing a systematic and accessible framework for establishing the existence of sustained oscillations in biologically relevant models.

Material and methods

The Original Three-Dimensional Model

To describe predator–prey interactions with greater biological realism, we consider an adaptive vulnerability predator–prey (AV-PP) model in which prey susceptibility to predation is treated as a dynamic variable. Unlike classical predator–prey models, this formulation incorporates adaptive mechanisms that allow the prey population to modify its vulnerability in response to predation pressure, providing a more realistic representation of ecological interactions ([1];[4]).

Let $x(t)$, $y(t)$, and $v(t)$ denote the prey population, predator population, and prey vulnerability, respectively, at time t. The dynamics of the system are governed by the following three-dimensional system of ordinary differential equations:

$$\begin{cases} \dot{X} = rx\left(1 - \frac{x}{k}\right) - avxy, \\ \dot{Y} = bavxy - dy, \\ \dot{V} = \gamma\left(\frac{x}{k} - v\right). \end{cases} \tag{1}$$

where all parameters are assumed to be positive to ensure biological feasibility. The inclusion of the vulnerability variable $v(t)$ allows the model to capture adaptive ecological mechanisms such as behavioral changes and phenotypic plasticity in prey populations. This extension represents a significant improvement over classical models, which assume constant predation rates and neglect adaptive responses.

Table 1. variables and parameters of the model.

Symbol	Description
X	Prey population.
Y	Predator population.
V	Predatory ability of prey.
r	Prey logistic growth rate.
k	Environmental capacity.
a	Predation rate.
b	Conversion efficiency of prey into predator biomass.
d	Predator mortality rate.
γ	Predation susceptibility adaptation rate.

Reduction to a Two-Dimensional System

Although the three-dimensional AV-PP model provides a biologically realistic description, its direct qualitative analysis is mathematically challenging. In particular, the application of fundamental tools for detecting periodic solutions, such as the Poincaré–Bendixson theorem, requires the system to be planar (two-dimensional), which is not satisfied by the original model ([8];[4]).

To overcome this limitation, we perform a systematic reduction based on time-scale separation. It is assumed that the prey vulnerability variable $v(t)$ evolves on a much faster time scale than the population variables $x(t)$ and $y(t)$. This assumption is biologically justified, as adaptive traits typically respond more rapidly than population densities [7].

Under this fast–slow framework, the quasi-steady-state approximation (QSSA) is applied to the third equation by setting:

$$\frac{dv}{dt} \approx 0$$

which yields the equilibrium relation:

$$v = v^*, \text{ where } v^* = {}^{x}\!/_{k}.$$

Substituting this expression into the first two equations of the original system, we obtain the following reduced planar model:

$$\begin{cases} \dot{x} = rx\left(1 - \frac{x}{k}\right) - \frac{a}{k}x^2 y, \\ \dot{y} = \frac{a}{k}bx^2 y - dy, \end{cases} \tag{2}$$

The system (2) represents the reduced two-dimensional predator–prey model obtained from the original formulation.

This reduction preserves the essential nonlinear interaction structure of the system while eliminating fast transient dynamics associated with the vulnerability variable. From a mathematical standpoint, the procedure is supported by singular perturbation theory, ensuring that the qualitative behavior of the system is maintained. From a biological perspective, it reflects the assumption that prey vulnerability adjusts instantaneously relative to population dynamics.

Stability Analysis of the Reduced Model

System (2) represents the reduced form of the original adaptive vulnerability predator–prey model, where the vulnerability variable has been approximated by $v = {}^{x}\!/_{k}$. The reduction allows the model to be analyzed in the phase plane while preserving the essential biological interaction between prey and predator populations.

To determine the equilibrium points, put

$$\frac{dx}{dt} = 0,$$

$$\frac{dy}{dt} = 0.$$

From the second equation:

$$y\left(\frac{a}{k}bx^2 - d\right) = 0,$$

For the positive coexistence equilibrium, $y > 0$,

$$\left(\frac{a}{k}bx^2 - d\right) = 0$$

which gives

$$x^* = \sqrt{\frac{dk}{ab}}$$

Substituting x^* into the first equation gives:

$$y^* = \frac{r(k - x^*)}{ax^*}$$

Thus, the positive equilibrium point is $E^* = (x^*, y^*)$.

The Jacobian matrix of system (2) is

$$J(f(x,y)) = \begin{bmatrix} r(1 - 2\frac{x}{k}) - 2\frac{a}{k}xy & -\frac{a}{k}x^2 \\ 2\frac{ab}{k}xy & \frac{ab}{k}x^2 - d \end{bmatrix}$$

Evaluating the Jacobian at the interior equilibrium E^*, gives

$$J(E^*) = \begin{bmatrix} r(1 - 2\frac{x^*}{k}) - 2\frac{a}{k}x^*y^* & -\frac{a}{k}x^{*2} \\ 2\frac{ab}{k}x^*y^* & \frac{ab}{k}x^{*2} - d \end{bmatrix}$$

$$J(E^*) = \begin{bmatrix} -r & -\frac{d}{b} \\ 2br(1 - \frac{x^*}{k}) & 0 \end{bmatrix},$$

The trace and determinant are therefore

$$tr\big(J(E^*)\big) = -r + 0 = -r < 0.$$

$$det\big(J(E^*)\big) = (-r).(0) - \left(-\frac{d}{b}\right).(2br\left(1 - \frac{x^*}{k}\right))$$

$$= 2dr\left(1 - \frac{x^*}{k}\right) > 0, \text{ since } x^* < k.$$

and hence the equilibrium point E^* is locally asymptotically stable.

trajectories tend toward the coexistence equilibrium rather than forming closed periodic orbits. This agrees with the phase-plane behavior shown in the original analysis of the reduced model, where trajectories converge to a stable equilibrium instead of producing sustained oscillations. The stability results obtained analytically are further confirmed by the phase-plane diagrams shown in Figures 1 and 2.

From a biological point of view, this means that the predator population responds proportionally and efficiently to changes in prey density. As a result, the interaction between prey and predator acts as a stabilizing mechanism, preventing persistent population cycles. This limitation motivates the next step of the study: replacing the linear predation term with nonlinear Holling functional responses to introduce more realistic ecological mechanisms such as predator saturation and handling time.

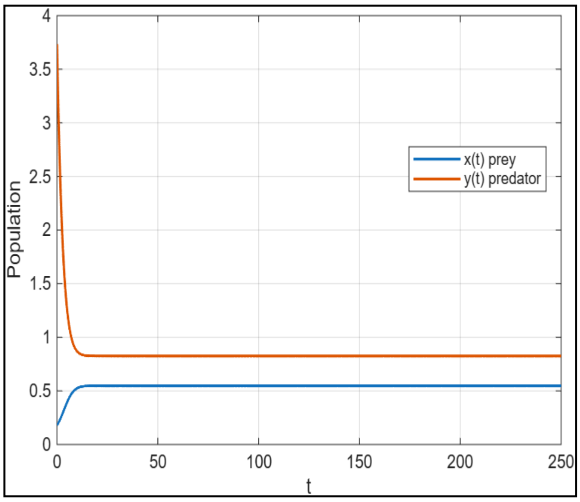

Figure 1. Stability of the equilibrium point in the system

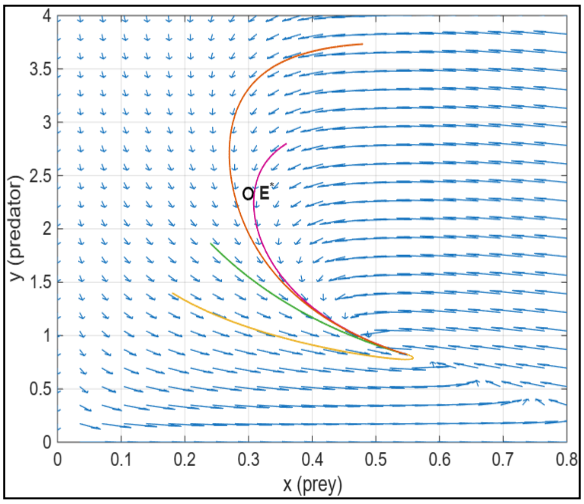

Figure 2. The Direction field and trajectories.

The direction field and trajectories clearly show that all solutions converge toward the interior equilibrium point regardless of the initial conditions within the biologically relevant region. No closed orbits or periodic trajectories are observed in the phase plane. Instead, all trajectories approach the equilibrium monotonically or through damped oscillations, indicating that the equilibrium is globally asymptotically stable. This graphical behavior is consistent with the analytical results obtained from the Jacobian matrix, confirming that the reduced system does not support limit cycle dynamics, moreover, resulting in a stable coexistence without sustained oscillations.

Transition to **Holling** Type II Model

The analysis of the reduced system (2) demonstrates that all trajectories converge toward a stable coexistence equilibrium, and no periodic oscillations arise. This result indicates that the system does not admit limit cycles under the linear interaction structure.

The interior equilibrium, as previously shown, ensures local asymptotic stability, while the boundedness of solutions prevents divergence. Consequently, the system lacks the necessary conditions for the emergence of sustained oscillatory behavior. Biological this behavior reflects an idealized interaction in which the predator responds instantaneously and proportionally to changes in prey density. Such a formulation assumes that the predation rate increases indefinitely with prey abundance, which is not realistic in natural ecosystems. In practice, predators are subject to physiological and behavioral constraints, such as handling time, satiation, and limited consumption capacity.

To overcome these limitations and introduce more realistic ecological dynamics, the linear predation term is replaced by a nonlinear functional response. In particular, consider the Holling Type II functional response, which incorporates predator saturation through handling time and provides a more accurate description of predation processes ([2];[3]).

Model with Holling Type II Functional **Response**

The modified predator–prey system is obtained by replacing the linear interaction term in system (2) with the Holling Type II functional response. The resulting system is given by:

$$\begin{cases} \dot{x} = rx\left(1 - \frac{x}{k}\right) - \frac{a}{k}\frac{1}{1+hx}xy, \\ \dot{y} = \frac{ab}{k}\frac{1}{1+hx}xy - dy, \end{cases} \tag{3}$$

where $h > 0$ represents the handling time of the predator.

The introduction of the Holling Type II functional response fundamentally alters the structure of the system. Unlike the linear model, the predation rate now saturates as prey density increases, reflecting the limited capacity of predators to consume prey.

Mathematically, the nonlinear term introduces a feedback delay between prey growth and predator response. This delay can destabilize the equilibrium under certain parameter conditions, creating the necessary mechanism for the emergence of limit cycles.

Biologically, this modification captures the effect of handling time, where predators require a finite amount of time to process each prey item. As prey density increases, the predator's consumption rate approaches a maximum limit, leading to a weakening of the regulatory effect on the prey population.

Application of the Elemental Limit Cycle (ELC) Method

To investigate the existence of limit cycles in the modified system (3), we apply the Elemental Limit Cycle (ELC) method, which provides a systematic framework for analyzing boundedness and oscillatory behavior in predator–prey models [10].

Step 1: Matching the System to the General ELC Form

The system (3) can be written in the general predator–prey form:

$$\dot{x} = xg(x) - \xi(y)p(x),$$

$$\dot{y} = \eta(y)[-\gamma + q(x)]$$

where

$$g(x) = r(1 - \frac{x}{k}), \xi(y) = y, p(x) = \frac{a}{k}\frac{x}{1+hx}, \eta(y) = y, \gamma = d, q(x) = bp(x).$$

These functions satisfy the standard structure required for the application of the ELC method.

Step 2: Verification of ELC Assumptions

- The prey growth functions:

(H1): $g(x) = r\left(1 - \frac{x}{k}\right), g(0) = r > 0, k > 0$ there exists a carrying capacity, such that $g(x) > 0$ for $x \in [0, k)$, $g(k) = 0$ and $g(x) < 0$ for $x > k$.

- The functional response:

(H2): $p(x)$ and $q(x)$ are differentiable, with $p(0) = q(0) = 0, \dot{p}(x) = \frac{a}{k}\frac{1}{(1+hx)^2}$, and $\dot{q}(x) = b\dot{p}(x), \forall x > 0$, and there exist positive numbers a_1 and a_2 such that $\frac{p(k)}{k}x \le p(x) \le a_1, \forall x \in [0, k]$ and $\frac{q(k)}{k}x \le q(x) \le a_2, \forall x \in [0, k]$.

- Predator related functions:

(H3): $\xi(y) = y$ and $\eta(y) = y$, implying $\xi(0) = \eta(0) = 0, \dot{\xi}(y) = 1 > 0$, and $\dot{\eta}(y) = 1 > 0, \forall y$, there exist positive constants s and S so that $sy \le \xi(y) \le Sy$ for all $y \ge 0$, and there exist positive constants n and N so that $-Ny \le -\gamma\eta(y) \le -ny$ for all y.

Step 3: Boundedness of Solutions
To apply the Elemental Limit Cycle (ELC) method, it is necessary to show that all solutions of system (3)
Boundedness of the Prey Population $x(t)$

The prey dynamics are governed by a logistic growth term. If $x(t) \geq k$, then $\dot{x} \leq 0$, which implies that $x(t)$ cannot remain above k for all time. Therefore,

$$0 < x(t) \leq k.$$

and hence the prey population is bounded.

- Boundedness of the Predator Population $y(t)$

To establish the boundedness of $y(t)$, consider the auxiliary function

$$\omega(t) = x(t) + \frac{1}{b} y(t) \tag{4}$$

Differentiating with respect to time gives

$$\dot{\omega}(t) = \dot{x}(t) + \frac{1}{b} \dot{y}(t)$$

Substituting from system (3), we obtain

$$\dot{\omega}(t) = rx\left(1 - \frac{x}{k}\right) - \frac{d}{b} y$$

Since $x(t) \in [0, k]$, the function $g(x) = r(1 - \frac{x}{k})$ attains its maximum at $x = \frac{k}{2}$, yielding

$$rx(1 - \frac{x}{k}) \leq \frac{rk}{4}$$

thus,

$$\dot{\omega} \leq \frac{rk}{4} - \frac{d}{b} y \tag{5}$$

Using the relation (4), gives

$$y(t) \leq b\omega(t)$$

Substituting into the inequality (5) yields

$$\dot{\omega}(t) \leq \frac{rk}{4} - d\omega(t)$$

which can be written as

$$\dot{\omega}(t) + d\omega(t) \leq \frac{rk}{4}$$

Solving this linear differential inequality gives

$$\omega(t) \leq \omega(0)e^{-dt} + \frac{rk}{4d}(1 - e^{-dt})$$

Taking the limit superior as $t \to \infty$, we obtain

$$\limsup_{t\to\infty} \omega(t) \leq \frac{rk}{4d}.$$

Thus, $\omega(t)$ is uniformly bounded for all $t \geq 0$. Since $x(t)$ is bounded by k and $\omega(t)$ is bounded, it follows directly that $y(t)$ is also bounded for all $t \geq 0$.

Step 4: Instability of the Interior Equilibrium

To complete the application of the ELC method, we analyze the stability of the interior equilibrium point of system (3). The positive equilibrium is denoted by $E^* = (\frac{dk}{ab-dhk}, \frac{rk}{a}(1+hx^*)\left(1-\frac{x^*}{k}\right))$

The Jacobian matrix of the system is

$$J\big(f(x,y)\big) = \begin{bmatrix} r\left(1-\frac{2x}{k}\right) - \frac{a}{k}\cdot\frac{y}{(1+hx)^2} & -\frac{a}{k}\cdot\frac{x}{1+hx} \\ \frac{ab}{k}\cdot\frac{1}{(1+hx)^2}y & \frac{ab}{k}\cdot\frac{1}{1+hx}x - d \end{bmatrix},$$

and hence

$$J(E^*) = \begin{bmatrix} r\left(1-\frac{2x^*}{k}\right) - \frac{a}{k}\cdot\frac{y^*}{(1+hx^*)^2} & -\frac{d}{b} \\ \frac{ab}{k}\cdot\frac{1}{(1+hx^*)^2}y^* & 0 \end{bmatrix}$$

let

$$J(E^*) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

it follows that

$$B < 0, C > 0.$$

which implies

$$det\big(J(E^*)\big) = -BC > 0$$

Thus, the sign of the trace

$$tr\big(J(E^*)\big) = A = r\left(1-\frac{2x^*}{k}\right) - \frac{a}{k}\cdot\frac{y^*}{(1+hx^*)^2}$$

determines the stability of the equilibrium point. Under suitable conditions on the system parameters, the trace becomes positive, that is,

$$tr\big(J(E^*)\big) > 0$$

In this case, the interior equilibrium E^*, is unstable (either an unstable node or focus). This instability indicates that trajectories starting near the equilibrium move away from it over time.

This result plays a crucial role in the qualitative analysis of the system, as the instability of the equilibrium, together with the boundedness of solutions established in the previous step, provides the necessary conditions for the existence of a limit cycle. This instability reflects the inability of the predator to immediately regulate the prey population due to the saturation effect introduced by the nonlinear functional response. As a result, the system does not settle at equilibrium, but instead tends toward oscillatory behavior.

Step 5: Existence of a Limit Cycle

Having established the boundedness of solutions in Step 3 and the instability of the interior equilibrium in Step 4, we are now in a position to determine the long-term dynamical behavior of system (3). From Step 3, all trajectories of the system remain confined within a closed and bounded region in the positive quadrant. Moreover, the solutions do not approach infinity and are contained in a positively invariant compact set. From Step 4, the interior equilibrium point E^* is unstable under suitable conditions on the system parameters, as indicated by the positivity of the trace of the Jacobian matrix. This implies that trajectories in a neighborhood of the equilibrium cannot converge to it.

Since system (3) is a two-dimensional dynamical system, and all its trajectories are bounded, while the only interior equilibrium is unstable, the conditions of the Poincaré–Bendixson theorem are satisfied ([8]; [4]). Therefore, the system must admit at least one periodic orbit in the phase plane, hence, it follows that the system possesses a limit cycle.

Furthermore, due to the boundedness of trajectories and the instability of the equilibrium, this limit cycle is stable and attracts nearby solutions. This indicates the presence of sustained oscillations in the populations of prey and predator. The emergence of a limit cycle reflects the combined effect of nonlinear predation and delayed predator response due to handling time. As prey density increases, predator consumption saturates, reducing the immediate regulatory effect on the prey population. This delay creates feedback dynamics that lead to persistent oscillations rather than convergence to a steady state.

To further illustrate the dynamical behavior of system, phase-plane diagrams are presented in Figures 3 and 4. These figures provide a graphical confirmation of the analytical results obtained using the ELC method.

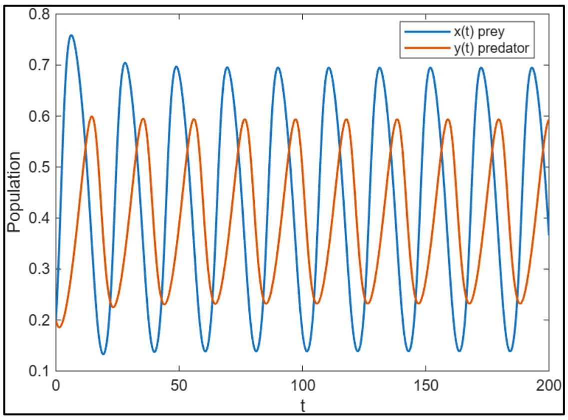

Figure 3. System Solution Oscillatory

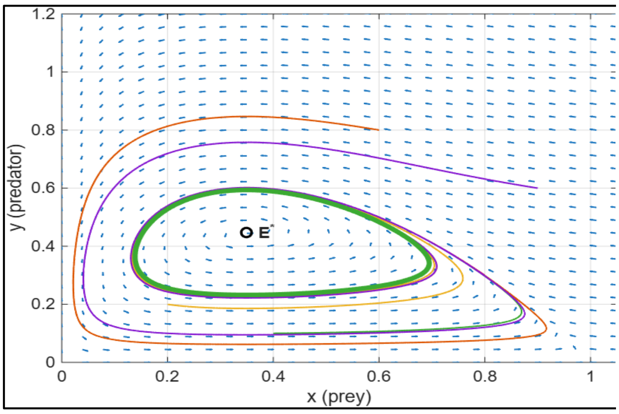

Figure 4. Limit Cycle Due to Holling Type II

Figures 3 and 4 illustrate the phase-plane dynamics of system (3). The trajectories clearly demonstrate the presence of a closed orbit surrounding the interior equilibrium point, confirming the existence of a limit cycle predicted by the analytical results. In Figure 3, the phase portrait reveals that the equilibrium point is unstable, as nearby trajectories move away from it and are attracted to a closed periodic orbit. Figure 4 further shows that solutions starting from different initial conditions converge toward the same closed trajectory, indicating that the limit cycle is stable.

These graphical results are in full agreement with the conclusions obtained from the application of the Poincaré–

Bendixson theorem and the ELC method, providing strong evidence of sustained oscillatory dynamics in the system.

Model with Holling Type III Functional Response

Extension to Holling Type III Functional Response

To further enhance the biological realism of the predator–prey interaction, system (3) is extended by incorporating a Holling Type III functional response. This formulation accounts for ecological mechanisms such as predator learning and prey refuge at low prey densities, which are not captured by the Holling Type II response [2].

The resulting system is given by

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{k}\right) - \dfrac{a}{k}\dfrac{1}{1 + hx^2}x^2y, \\ \dfrac{dy}{dt} = \dfrac{ab}{k}\dfrac{1}{1 + hx^2}x^2y - dy, \end{cases} \tag{6}$$

System (6) preserves the general structure of the previous model while modifying the functional response. In particular, the predation term now exhibits a sigmoidal dependence on prey density. At low prey densities, the predation rate increases slowly, reflecting reduced predator efficiency or prey refuge. As prey density increases, the predation rate accelerates before eventually saturating due to handling limitations.

Analytical Justification

The analytical procedure developed for system (3) can be extended to system (6) due to the similarity in the structure of the functional response. In particular, the assumptions required for the application of the Elemental Limit Cycle (ELC) method remain satisfied.

Indeed, the prey population is still governed by logistic growth and therefore remains bounded. Moreover, the functional response in system (6) is positive, differentiable, and bounded, ensuring that the corresponding predator dynamics also remain bounded using arguments analogous to those in Step 3. In addition, the Jacobian matrix evaluated at the interior equilibrium retains a positive determinant, while its trace may become positive under suitable parameter conditions, implying instability of the equilibrium.

Existence of Limit Cycle

Since system (6) is planar, its solutions are bounded, and the interior equilibrium can be unstable, the conditions of the Poincaré–Bendixson theorem are satisfied ([8]; [4]). Therefore, system admits at least one limit cycle. The dynamical behavior of system is illustrated in Figures 5 and 6.

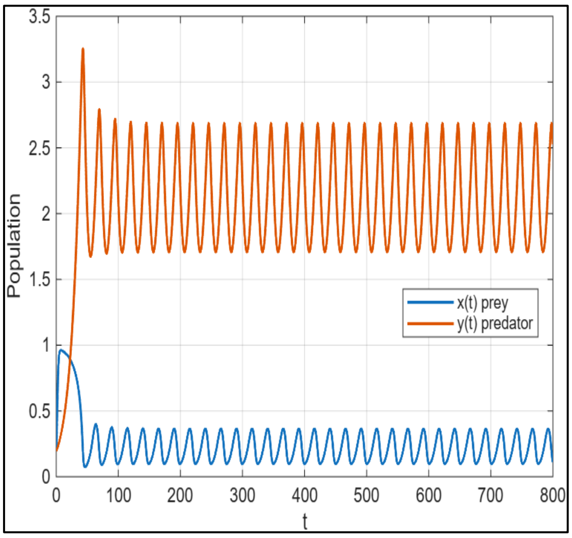

Figure 5. System Solution Oscillatory

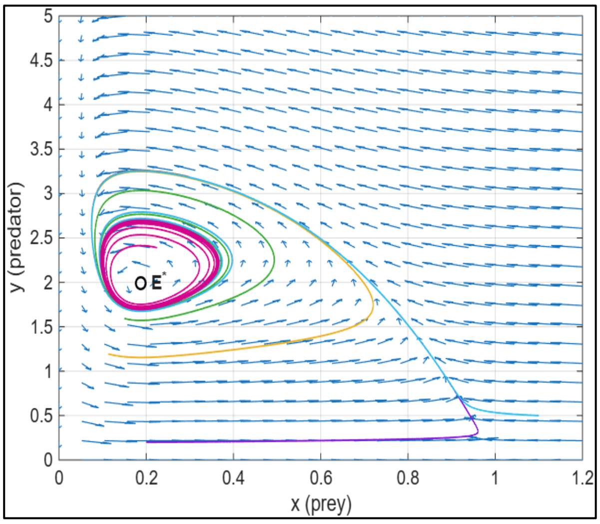

Figure 6. Limit Cycle under a Holling type III

The Holling Type III functional response leads to more realistic and structured oscillatory behavior compared to the previous models. In particular, the reduced predation pressure at low prey densities and the gradual increase in predator efficiency contribute to the persistence and stability of population cycles**.**

Results and discussion

The results obtained in this study reveal a clear progression in the dynamical behavior of predator–prey systems as the structure of the functional response is modified.

For the reduced model (2), the analysis shows that the system converges to a stable interior equilibrium. Both the Jacobian analysis and phase-plane diagrams confirm the absence of periodic solutions. All trajectories approach the equilibrium point, indicating that the system does not admit limit cycles. This behavior reflects a stabilizing interaction between predator and prey populations, even after incorporating adaptive vulnerability through model reduction.

In contrast, the introduction of the Holling Type II functional response in system (3) significantly alters the system dynamics. The inclusion of handling time introduces a nonlinear saturation effect, which destabilizes the equilibrium under suitable parameter conditions. As a result, the system exhibits a stable limit cycle. The analytical findings obtained via the ELC method are supported by the graphical results, which clearly show closed trajectories attracting nearby solutions.

Further extension to the Holling Type III functional response in system (6) enhances the biological realism of the model. The sigmoidal form of the functional response captures ecological mechanisms such as predator learning and prey refuge at low prey densities. While the system still exhibits limit cycle behavior, the oscillations are more structured and robust compared to system (3), particularly in regions of low prey density.

Overall, the comparison between the three models highlights a fundamental transition in system dynamics. The reduced model (2) exhibits stable equilibrium behavior, whereas systems (3) and (6) display sustained oscillations. This demonstrates that model reduction alone is not sufficient to generate periodic dynamics; rather, the emergence of limit cycles is primarily driven by the nonlinear structure of the functional response.

From a biological perspective, these results emphasize the importance of incorporating realistic predator behavior into mathematical models. Linear interactions fail to capture essential ecological processes, while nonlinear functional responses introduce the feedback mechanisms necessary for sustained population oscillations.

Conclusion

This study investigated the dynamical behavior of an adaptive predator–prey model through model reduction and the incorporation of nonlinear functional responses. The reduced model was shown to exhibit stable equilibrium behavior, while the introduction of Holling Type II and Type III functional responses led to the emergence of sustained oscillations in the form of limit cycles.

The results demonstrate that model reduction provides an effective framework for simplifying complex systems without altering their qualitative dynamics, whereas nonlinear functional responses play a decisive role in generating oscillatory behavior and improving biological realism.

These findings contribute to a better understanding of ecological population dynamics and highlight the importance of incorporating realistic interaction mechanisms in mathematical models. This study contributes to improving ecological management and understanding real-world population interactions, and future work may focus on extending the model to stochastic environments or spatial dynamics.

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