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On Some Fixed Point Results in Complete 2-Metric Spaces

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نتائج حول بعض النقاط الثابتة في فضاءات مترية ثنائية كاملة

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Abstract:

The present paper proves a number of related results concerning fixed points in the area of complete 2-metric spaces. These results represent versions of several fixed-point theorems in complete metric spaces that were established in the literature at a later stage. By extending classical results from metric spaces to the more general framework of 2-metric spaces, we aim to provide deeper insight into the structure and behavior of mappings defined on such spaces.

In particular, we investigate conditions under which the existence and uniqueness of fixed points can be guaranteed. The obtained results not only generalize well-known theorems but also contribute to the ongoing development of fixed-point theory in generalized metric settings.

Keywords: Fixed Points, 2- Metric Spaces, Complete 2- Metric Spaces.

الملخص

تُثبت هذه الورقة البحثية عددًا من النتائج المرتبطة بنقاط الثابتة في مجال الفضاءات مترية ثنائية كاملة. وتمثل هذه النتائج صيغًا لبعض مبرهنات نقطة الثابتة في الفضاءات المتريّة الكاملة، والتي أُثبتت في الأدبيات العلمية في مرحلة لاحقة. ومن خلال توسيع النتائج الكلاسيكية من الفضاءات المتريّة إلى الإطار الأكثر عمومية للفضاءات مترية ثنائية كاملة، نهدف إلى تقديم فهم أعمق لبنية وسلوك التطبيقات المعرفة على مثل هذه الفضاءات. وعلى وجه الخصوص، نبحث في الشروط التي يمكن من خلالها ضمان وجود النقاط الثابتة ووحدانيتها. ولا تقتصر النتائج المتحصَّل عليها على تعميم المبرهنات المعروفة فحسب، بل تسهم أيضًا في التطور المستمر لنظرية النقطة الثابتة في الأطر المتريّة المعممة.

الكلمات المفتاحية: النقاط الثابتة، فضاءات القياس الثنائية، فضاءات القياس الثنائية الكاملة.

Introduction

The concept of 2-metric spaces was introduced by Gahler in [1], [7]. It has been shown by Gahler that in 2-metric d is non-negative real-valued function. The fixed-point theory in 2- metric space is one of the most fundamental and important subjects in modern mathematics which is widely applied in many other branches of applied science .

The fixed-point theory in 2- metric space has been developed extensively in different subjects by others, for examples [1], [3], [4], [5], [6] and [9].

The present paper deals with fixed point theory in the setting of complete 2-metric spaces.

We begin with the following fundamental definition.

Definition 1

Let T be a mapping from a non-empty set X into itself such that $T(x) = x$. Then x in X is called a fixed point of T. Let T_1 and T_2 be two mappings from X into itself. If there exists an element x in X such that $T_1(x) = T_2(x) = x$, then x is called a *common fixed point* of T_1 and T_2.

Let $x_0 \in X$ and set $x_1 = T(x_\mathbf{0})$. For this x_1, there exists x_2 such that $x_2 = T(x_\mathbf{1})$. Continuing this way, we can define

$$x_{n+1} = T(x_n) \quad (n = 0, 1, 2, ...),$$

or

$$x_n = T(x_{n-\mathbf{1}}) \quad (n = 1, 2, ...).$$

In the following , we recall some definitions related to 2-metric spaces, introduced in [4].

Definition 2

Let X be a non-empty set and let $d : X \times X \times X \to \mathbb{R}$ be a mapping satisfying the following conditions :

(i) for any two distinct points $x, y \in X$ there exists a point $z \in X$ such that $, d(x, y, z) \neq 0$

(ii) $d(x, y, z) = 0$ if at least two of three points x, y, z are equal ,

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,

(iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$,

for all $x, y, z, u \in X$.

The mapping d is called a *2-metric* on X and the pair (X, d) is called a *2-metric space*

Remark

Note that every 2-metric contains at least three points .

Definition 3

A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a *convergent sequence* to a point x in

X if $\lim_{n \to \infty} d(x_n, x, z) = 0$ for all $z \in X$.

The point x is called the *limit* of the sequence $\{x_n\}$ in X.

The limit of a sequence in a 2-metric space, if exists, is unique.

Lemma 1 [10]

Let $\{x_n\}$ be a sequence in a 2 - metric space (X, d). If $\{x_n\}$ converges to a point x in X, then $\lim_{n \to \infty} d(x_n, y, z) = d(x, y, z)$ for all $x, y, z \in X$.

Definition 4

A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a *Cauchy sequence* in X if $\lim_{n, m \to \infty} d(x_n, x_m, z) = 0$ for all $z \in X$.

Definition 5

Let (X, d) be a 2 - metric space. If every Cauchy sequence in X is convergent in X, then (X, d) is called a *complete 2-metric space*.

Example 1 [2]

Let X be a non-empty set and let $x, y, z \in X$. Define

$$d(x, y, z) = \begin{cases} 0 & \text{if } x, y, z \text{ are not distinct} \\ 1 & \text{otherwise}. \end{cases}$$

Then (X, d) is a complete 2-metric space .

Lemma 2 [8]

Let $\{x_n\}$ be a sequence in a complete 2 - metric space (X, d). If there exists $h \in (0,1)$ such that $d(x_n, x_{n+1}, z) \leq h\, d(x_{n-1}, x_n, z)$ for all $z \in X$, then $\{x_n\}$ converges to a point in X.

Main Results

In this section, we explore fixed point results for mappings defined on 2-complete metric spaces. In particular, we examine sufficient conditions under which a self-mapping admits a fixed point and discuss how these results relate to classical fixed-point theorems.

Theorem 3

Let T be a mapping from a complete 2-metric space (X, d) into itself satisfying

$d(T(x), T(y), z) \leq a\big(d(T(x), x, z) + d(T(y), y, z)\big) + b\, d(x, y, z)$,

for all $x, y, z \in X$ and $a \geq 0, b \geq 0$ with $2a + b < 1$.

Then T has a unique fixed point in X.

Proof

Let $x_0 \in X$. We define the sequence $\{x_n\}$ by

$$x_{n+1} = T(x_n) \quad (n = 0, 1, 2, \ldots).$$

If $x_{n+1} = x_n$, then T has a fixed point in X.

Assume that $x_{n+1} \neq x_n$. Then

$$\begin{aligned} d(x_n, x_{n+1}, z) &= d(T(x_{n-1}), T(x_n), z) \\ &\leq a\big(d(T(x_{n-1}), x_{n-1}, z) + d(T(x_n), x_n, z)\big) \\ &\quad + b\, d(x_{n-1}, x_n, z) \\ &= a\big(d(x_n, x_{n-1}, z) + d(x_{n+1}, x_n, z)\big) + b\, d(x_{n-1}, x_n, z). \end{aligned}$$

Therefore

$$(1-a)\,d(x_n, x_{n+1}, z) \leq (a+b)\,d(x_{n-1}, x_n, z),$$

and so

$$d(x_n, x_{n+1}, z) \leq \frac{a+b}{1-a}\, d(x_{n-1}, x_n, z)$$

$$, d(x_n, x_{n+1}, z) \leq h\, d(x_{n-1}, x_n, z)$$

where $h = \frac{a+b}{1-a} < 1$.

Hence in view of Lemma 2 **,** the sequence $\{x_n\}$ converges to the element in X , say u .

That is , $\lim_{n \to \infty} d(x_n, x, z) = 0$ for all $z \in X$.

We have

$$d(T(u), x_n, z) = d(T(u), T(x_{n-1}), z)$$

$$\leq a\big(d(T(u), u, z) + d(T(x_{n-1}), x_{n-1}, z)\big) + b\, d(u, x_{n-1}, z)$$

$$= a\big(d(T(u), u, z) + d(x_n, x_{n-1}, z)\big) + b\, d(u, x_{n-1}, z).$$

Letting $n \to \infty$. Then $(1-a)\, d(T(u), u, z) \leq 0$.

Since $1-a > 0$, it follows that $d(T(u), u, z) = 0$.

Thus $T(u) = u$ and hence T is a fixed point in X .

For uniqueness, let v be another fixed point of T . Then

$$d(u, v, z) = d(T(u), T(v), z)$$

$$\leq a\,(d(T(u), u, z) + d(T(v), v, z)) + b\, d(u, v, z).$$

Therefore $d(u, v, z) \leq b\, d(u, v, z)$ and so $(1-b)\, d(u, v, z) \leq 0$.

Since $1-b > 0$, it follows that $d(u, v, z) = 0$. Therefore $u = v$.

This completes the proof.

The following corollaries follow as a consequence of Theorem 3**.**

Corollary 4

Let (X, d) be a complete 2- metric space and let T be a mapping from X into itself satisfying

$$d(T(x), T(y), z) \leq \lambda \big(d(T(x), x, z)\,.\,d(T(y), y, z)\,.\,d(x, y, z)\big)^{\frac{1}{3}},$$

for all $x, y, z \in X$ and $0 < \lambda < 1$.

Then T has a unique fixed point in X .

Proof

Let $x, y, z \in X$. We have

$$d(T(x), T(y), z) \leq \lambda \big(d(T(x), x, z)\,.\,d(T(y), y, z)\,.\,d(x, y, z)\big)^{\frac{1}{3}}$$

$$\leq \lambda \left(\frac{d(T(x), x, z)\,.\,d(T(y), y, z)\,.\,d(x, y, z)}{3} \right).$$

The statement is implied by Theorem 3 for $\alpha = \beta = \dfrac{\lambda}{3}$.

Corollary 5

Let (X, d) be a complete metric space and let T be a mapping from X into itself satisfying

$$d(T(x), T(y), z) \leq \alpha \frac{d(T(x), x, z)^2 + d(T(y), y, z)^2}{d(T(x), x, z) + d(T(y), y, z)} + \beta\, d(x, y, z),$$

for all $x, y, z \in X$ and $\alpha \geq 0, \beta \geq 0, 2\alpha + \beta < 1$.

Then T has a unique fixed point in X.

Proof

Let $x, y, z \in X$. We have

$$d(T(x), T(y), z) \leq \alpha \frac{d(T(x), x, z)^2 + d(T(y), y, z)^2}{d(T(x), x, z) + d(T(y), y, z)} + \beta\, d(x, y, z)$$

$$\leq \alpha \left(d(T(x), x, z) + d(T(y), y, z) \right) + \beta\, d(x, y, z).$$

Hence T admits a unique fixed point by Theorem 3.

Corollary 6

Let T_k $(k = 1, 2, \ldots, n)$ be a family of mapping from a complete 2 - metric space (X, d) into itself such that $T_1 T_2 \ldots T_n$ commute with T_i $(i = 1, 2, \ldots, n)$ satisfying

$d((T_1 T_2 \ldots T_n)(x), (T_1 T_2 \ldots T_n)(y), z)$ £ $a(d((T_1 T_2 \ldots T_n)(x), x, z)$
$+ d((T_1 T_2 \ldots T_n)(y), y, z)) + b\, d(x, y, z),$

for all x, y, z خ X and a ³ 0, b ³ 0 with $2a + b < 1$.

Then T_k have a unique common fixed point in X.

Proof

Set $V = T_1 T_2 \ldots T_n$. Then

$d(V(x), V(y), z)$ £ $a(d(V(x), x, z) + d(V(y), y, z)) + b\, d(x, y, z)$.

Then by Theorem 3, V has a unique fixed point in X, say x¢.

So $V(x$¢$) = x$¢. Then

$$V(T_i(x¢)) = T_i(V(x¢)) \quad (i = 1, 2, \ldots, n)$$
$$= T_i(x¢).$$

Thus $T_i(x$¢$)$ is a fixed point of V. Since V has a unique fixed point x¢, it follows that $T_i(x$¢$) = x$¢. Thus x¢ is a common fixed point of T_i.

For uniqueness, let x¢and x¢¢ be fixed points of T_i.

Then

$$V(x'') = (T_1 T_2 \dots T_n)(x'') = (T_1 T_2 \dots T_{n-1})(T_n(x''))$$

$$= (T_1 T_2 \dots T_{n-1})(x'') = (T_1 T_2 \dots T_{n-2})(x''),$$

and so, we can obtain $V(x'') = x''$.

Since V has a unique fixed point x', it follows that $x' = x''$.

This completes the proof.

Theorem 7

Let T_1 and T_2 be mappings from a complete 2-metric space (X, d) into itself satisfying

$d(T_1(x), T_2(y), z) \le a\big(d(T_1(x), x, z) + d(T_2(y), y, z)\big) + b\, d(x, y, z)$,

for all $x, y, z \in X$ and $a \ge 0$, $b \ge 0$ with $2a + b < 1$.

Then T_1 and T_2 have a unique common fixed point in X.

Proof

Let $x_0 \in X$. Define

$$x_{2n+1} = T_1(x_{2n}) \quad (n = 0, 1, 2, \dots),$$

and

$$x_{2n+2} = T_2(x_{2n+1}) \quad (n = 0, 1, 2, \dots).$$

We have

$$d(x_{2n+1}, x_{2n}, z) = d(T_1(x_{2n}), T_2(x_{2n-1}), z).$$

In the same manner as in Theorem 3, we obtain

$$d(x_{2n+1}, x_{2n}, z) \le \frac{a+b}{1-a}\, d(x_{2n}, x_{2n-1}, z).$$

The latter implies that $d(x_{n+1}, x_n, z) \le \dfrac{a+b}{1-a}\, d(x_n, x_{n-1}, z)$

$$d(x_{n+1}, x_n, z) \le h\, d(x_n, x_{n-1}, z),$$

where $h = \dfrac{a+b}{1-a} < 1$.

So, the sequence $\{x_n\}$ is convergent to an element u in X. That is, $\lim_{n \to \infty} x_n = u$.

Using an argument similar to that in Theorem 3, we conclude that $d(T_1(u), u, z) = 0$.

Thus $T_1(u) = u$. Hence u is a fixed point of T_1.

Similarly, we can prove that u is a fixed point of T_2.

Hence u is a common fixed point of T_1 and T_2.

The proof of the uniqueness of T_1 and T_2 is analogous to the proof of uniqueness in Theorem 3.

As a consequence of Theorem 7, we obtain the following corollaries.

Corollary 8

Let (X, d) be **a** complete 2-metric space and let T_1, T_2 be mappings from X into itself satisfying

$$d(T_1(x), T_2(y), z) \leq \lambda \left(d(T_1(x), x, z) \,.\, d(T_2(y), y, z) \,.\, d(x, y, z)\right)^{\frac{1}{3}}$$

for all $x, y, z \in X$ and $0 < \lambda < 1$.

Then T_1 and T_2 have a unique common fixed point in X.

Proof

The proof is analogous to the proof of Corollary 4 in regards to T_1 and T_2.

Corollary 9

Let (X, d) be **a** complete 2-metric space let T_1, T_2 be mappings from X into itself satisfying

$$d(T_1(x), T_2(y), z) \leq \alpha \frac{d(T_1(x), x, z)^2 + d(T_2(y), y, z)^2}{d(T_1(x), x, z) + d(T_2(y), y, z)} + \beta\, d(x, y, z)$$

for all $x, y, z \in X$ and $\alpha \geq 0, \beta \geq 0, 2\alpha + \beta < 1$.

Then T_1 and T_2 have a unique common fixed point in X.

Proof

The proof is analogous to the proof of Corollary 5 in regards to T_1 and T_2.

Corollary 10

Let $p, q \in$ and T_1^p, T_2^q be mappings from a complete 2-metric space (X, d) into itself satisfying

$$d(T_1^p(x), T_2^q(y), z) \leq a\left(d(T_1^p(x), x, z) + d(T_2^p(y), y, z)\right)$$
$$+ b\, d(x, y, z)$$

for all x, y, z ﺧ X and a ³ $0, b$ ³ 0 with $2a + b < 1$.

Then T_1 and T_2 have a unique common fixed point in X.

Proof

According to Theorem 7, the mappings T_1^p and T_2^q have a unique common fixed point u in X. Thus $T_1^p(u) = u$. We have

$$T_1(u) = T_1(T_1^p(u)) = T_1^p(T_1(u)).$$

That is , $T_1(u)$ *is* a fixed point of T_1^p .

Similarly , $T_2^q(u) = u$. So

$$T_2(u) = T_2(T_2^q(u)) = T_2^q(T_2(u)).$$

That is , $T_2(u)$ *is* a fixed point of T_2^q . However, T_1^p and T_2^q have a unique fixed point . Therefore $T_1(u) = u$ and $T_2(u) = u$.

Thus u is a unique common fixed point of T_1 and T_2.

Conclusion

By utilizing the structure of complete 2-metric spaces, we obtained generalizations of fixed-point principles under some suitable conditions. The results presented here demonstrate that completeness in the sense of 2-metrics plays a central role in guaranteeing the existence and uniqueness of the results. Moreover, we analyzed the behavior of Cauchy sequences and convergence in complete 2-metric spaces and clarified their relationship with classical metric completeness. Our findings contribute to the growing body of literature on generalized metric structures and open new directions for further research. In particular, future work may focus on investigation of common fixed points for families of mappings.

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