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Genereliaed Upper and Lower Approixmations for Anti-Topology and their Structral Properties

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تعميم التقريبات العليا والسفلى على مضاد التوبولوجيا وخصائصه البنيوية

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Abstract:
This paper aims to study roughness in Ant-topology. In addition, we performed a comparison with approximations in classical topology. We found that the boundaries become wider as ambiguity increases, reflecting a more accurate model of uncertainty and approximation in complex. Moreover, we explain how effect of rough approximation on the result of ant-topology by creating more space for ambiguity and approximation, allowing for better modeling accuracy of phenomena characterized by high uncertainty, despite increased mathematical complexity.

Keywords: Upper Approximation, Lower Approximation, Anti-Topology, Rough Set.

الملخص
تهدف هذه الورقة البحثية إلى دراسة الخشونة في الطوبولوجيا المضادة. بالإضافة إلى ذلك، أجرينا مقارنة مع التقريبات في الطوبولوجيا الكلاسيكية. وجدنا أن الحدود تتسع مع ازدياد الغموض، مما يعكس نموذجًا أكثر دقة لعدم اليقين والتقريب في المركبات. علاوة على ذلك، شرحنا تأثير التقريب الخشن على نتيجة الطوبولوجيا المضادة من خلال توفير مساحة أكبر للغموض والتقريب، مما يسمح بدقة نمذجة أفضل للظواهر التي تتسم بعدم يقين كبير، على الرغم من زيادة التعقيد الرياضي.

الكلمات المفتاحية: التقريب من اعلي، التقريب من أسفل، مضاد الطوبولوجيا، المجموعة الخشنة.

1-Introduction

Pawlak [1] in 1982 introduced the rough set theory. It was a good formal tool for modeling and processing in information system. Algebraic structures of rough sets have been studied by many authors, see [2] , [3], [4]. The upper approximation of a given set is the union of all the equivalence classes that are subsets of the set, and the upper approximation is the union of all the equivalence classes that are intersection with set non-empty. Anitha K and Venkatesan study the rough Sets on Topological Spaces I [5], S¸ahin, Kargın and M. Y¨ucel have defined anti-topological spaces in [6] . Abd El-Monsef, and others introduce the near approximations in topological spaces[7]. The main purpose of this paper is to introduce the rough upper and lower approximations on anti-topology. In addition, we introduce some properties of approximations and these algebraic structures.

2-Preliminaries

We start by given some definitions and results about rough sets.

Suppose that ~ an equivalence relation on an universe set ($U \neq \emptyset\,, finite\ set$). Some authors say ~ is indiscernibility relation. The pair (U, ~) is called an approximation space. We use $U/\sim$ to denote the family of all equivalent classes$[x]_\sim$. The empty set $\varnothing$ and the element of $U/\sim$ are called elementary sets. For any $X \subseteq U$, we write X^c to denote the complementation of X in U.

Definition 2.1: Let (U, ~) be an approximation space. We define the upper approximation of X by $\overline{\sim X} = \{x \in U: [x]_\sim \cap X \neq \emptyset\}$ and the lower approximation of X by$\underline{\sim X} = \{x \in U: [x]_\sim \subseteq X\}$ the boundary is X = $\overline{\sim X} - \underline{\sim X}$. If B = $\emptyset$, we say X is exact (crisp) set otherwise, we say X is Rough set (inexact).

Preposition 2-1:

1) $\underline{\sim X} \subseteq X \subseteq \overline{\sim X}$
2) $\underline{\sim \emptyset} = \overline{\sim \emptyset}, \underline{\sim U} = \overline{\sim U}$,
3) $\underline{\sim(X \cup Y)} \supseteq \underline{\sim(X)} \cup \underline{\sim(Y)}$,
4) $\underline{\sim(X \cap Y)} = \underline{\sim(X)} \cap \underline{\sim(Y)}$,
5) $\overline{\sim(X \cup Y)} = \overline{\sim(X)} \cup \overline{\sim(Y)}$.
6) $\overline{\sim(X \cap Y)} \subseteq \overline{\sim(X)} \cap \overline{\sim(Y)}$.
7) $\overline{\sim X^C} = \left(\underline{\sim X}\right)^c$.
8) $\underline{\sim X^C} = (\overline{\sim X})^c$.
9) $\underline{\sim(\underline{\sim X})} = \overline{\sim(\underline{\sim X})} = \underline{\sim X}$. -
10) $\overline{(\sim(\overline{\sim X})} = \underline{\sim(\overline{\sim X})} = \overline{\sim X}$.

Now, we introduce some concepts of anti-topology. For more details see [9],[14].

Definition 2.2. Suppose that $(X \neq \emptyset)$ $universe$ and$\tau = \{$ A: A$\subseteq$ X$\}$. We call that (X, τ) is an anti-topological space if the following conditions are satisfied:

(1) $\emptyset, X \notin \tau$.
(2) $If\ A_1, A_2, .., A_n \in \tau, then\ \bigcup_{i=1}^{n} A_i \notin \tau, \forall\, n \in N.$
(3) $For\ any\ collection\ \{A_i\}_{i \in J \neq \emptyset}\ such\ that\ A_i \in \tau\ for\ each\ i \in J, \bigcup_{i \in J} A_i \in \tau$.

Definition 2.3. The elements of τ is called an anti-open sets. In addition, we call anti-closed sets if in the complements of anti-open and denote to the set of all anti-closed sets byτ_{cl}.

Definition 2.4. Suppose that (X, τ) be an anti-topological space and$A \subseteq X$. Then we define:-

1) An anti-interior of A by $AntiInt(A) = \cup\{O; O \subseteq A\ and\ O \in \tau\}$;
2) An anti-closure of A by $antiCl(A) = \cap\{F; A \subseteq F\ and\ F \in \tau_{cl}\}$

Example 2.1. $Suppose\ that\ X = \{1,2\} and\ \tau = \big\{\{1\},\{2\}\big\} = \tau_{cl}$.
$Clearly, \{1\} \cap \{2\} = \emptyset \notin \tau_{cl}\ and\ \{1\} \cup \{2\} = X \notin \tau_{cl}$.

Example 2.2. Suppose that $X = \{a,b,c,d\}\ and\ \tau = \{\{a,b\},\{b,c\},\{c,d\}\}$.
The intersections are$\{a,b\} \cap \{b,c\} = \{b\} \notin \tau$, $\{a,b\} \cap \{c,d\} = \emptyset \notin\ and\ \{b,c\} \cap \{c,d\} = \{c\} \notin \tau$.
The unions are$\{a,b\} \cup \{b,c\} = \{a,b,c\} \notin \tau, \{a,b\} \cup \{c,d\} = \{a,b,c,d\} = X \notin \tau\ and\ \{b,c\} \cup \{c,d\} = \{b,c,d\} \notin \tau$. We have τ_{cl}= $\{\{a,b\},\{a,d\},\{c,d\}\}$.
Note that $\{a,b\}\ and\ \{c,d\}$ are both anti-open and anti-closed. As for the associated space, it is $\tau_\tau = \{\emptyset, X, \{a,b\},\{b,c\},\{c,d\},\{b\},\{c\},\{a,b,c\},\{b,c,d\}\}$

Example 2.3. Suppose that $X = N^+$ and $\tau_k = \{only\ finite\ subsets\ of\ X\ which\ have\ cardinality\ k (k \in N^+)$.
Now, $if\ A,B \in \tau_k\ and\ A \neq B$. Clearly, $\emptyset \notin \tau_k\ and\ X = N \notin \tau_k$.

Example 2.4. Suppose that $X = R\ and\ T_y = \{[a,b], where\ b - a = y(\ y\ is\ a\ fixed\ positive\ real\ number)$. Now, $if\ A,B \in Ty\ and\ A \neq B, then\ their\ union\ has\ length\ z > y\ (moreover, it\ is\ possible\ that\ it\ is\ not\ an\ interval\ at\ all)\ and\ their\ intersection\ has\ length\ w < y\ (moreover, it\ can\ be\ empty\ or\ consist\ of\ one\ point). Clearly, \emptyset\ and\ X = R\ do\ not\ belong\ to\ Ty$.

Remarks 1.1 If (X, τ) is an anti-topological space, $B \in \tau\ and\ A \subseteq B$. Then$A \notin \tau$.

Example 2.5 Suppose that $X = \{1,2,3,4,5\}$ and $\tau = \{\{1,2\},\{3,4\},\{5\}\}$. then τ is an anti-topology on X .Consider Let$A = \{1,2,3\}$. Now $aint(A) = \{1,2\}\ and\ antiCl(A) = \{1,2,3,4\}$.

Proposition 2.1. Suppose that (X, τ) is an anti-topological space and$A, B \subseteq X$. Then $aint(A \cap B) \subseteq aint(A) \cap aint(B)$.

Proof. If$A \cap B \subseteq A, A \cap B \subseteq B$. From the monotonicity of interior we get that $aint(A \cap B) \subseteq aint(A)\ and\ aInt(A \cap B) \subseteq aInt(B)$. Hence $aint(A \cap B) \subseteq aint(A) \cap aint(B)$.

Remarks 1.2. The converse is not necessarily true.

Example 2.6. Suppose that $X = \{1,2,3,4,5\}, \tau = \big\{\{1,3\},\{2\},\{3,4\}\big\}, A = \{1,2,3\} and\ B = \{2,3,4\}$. we have $aint(A) = \{1,3\} \cup \{2\} = \{1,2,3\} and\ aint(B) = \{2\} \cup \{3,4\} = \{2,3,4\}$.
$Moreover, A \cap B = \{2,3\}\ and\ aint(A\ B) = \{2\}. Now\ aint(A) \cap aint(B) = \{2,3\} \not\subset \{2\}$

Some properties.
I n this section, we introduce some properties
Proposition 2.2 Suppose that $(X \neq \emptyset)$ and $U = \{A: \subseteq X\ that\ is\ anti-closed\ under\ finite\ intersections.$ Then 1- it is anti-closed under arbitrary intersections.
2- If it is anti-closed under arbitrary intersections, then anti-closed under arbitrary unions.
3- If it is anti-closed under arbitrary unions, then it is anti-closed under finite intersections.
Proposition. **2.3** Suppose (X,τ) is an anti-topological space and $A, B \in \tau_{cl}$. Suppose that $A \neq B$. Then $A \cap B \not\in \tau_{cl}$. and $\{A_i\}_{i\in J} \subseteq \tau_{cl}$. Then $\bigcup_{i\in J} A_i \not\in A_i$.

Definition 2.5. Suppose that (X,τ) is an anti-topological space. We called it is door anti-topological space if and only if each subset $(different\ than\ \emptyset\ and\ X)$ is anti-open or anti-closed.
Example 2.6. Suppose that $X = \{a,b,c\}, T = \{\{a\},\{b\},\{c\}\}$.
Remarks 1.3 If (X,τ) is an anti-topological space such that $|X| > 3 \Rightarrow (X,\tau)$ cannot be door space.
Definition 2.6.Suppose that $(X,\tau_1) and\ (Y,\tau_2)$ are two anti-topological spaces. The function $f : X \rightarrow Y$ is called anti-continuous if and only if $for\ any O \in \tau_2, f^{-1}(O) \in \tau_1$.
Example 2.7 Suppose that $X = \{1,2,3,4\}, \tau_1 = \{\{1,2\},\{3\}\}, Y = \{a,b,c,d,e\}, \tau_2 = \{\{a,b,c,d\},\{e\}\}$.
$Suppoae\ that\ f(1) = a, f(2) = b\ and\ f(3) = e.$
$Now\ f^{-1}(\{a,b,c,d\}) = \{a,b\} \in \tau_1 and\ f^{-1}(\{e\}) = \{3\} \in \tau_1$.
Definition 2.7. Suppose $X \neq \emptyset$ and $\tau = \{$ A: A⊆X$\}$. We call that (X,τ) is an neutro-topological space if the following conditions are satisfied:

1) $\emptyset \in \tau\ and\ X \notin \tau\ or\ X \in \tau\ and\ \emptyset \notin \tau$.
(4) $For\ at\ least\ n\ elements$ $A_1, A_2, .., A_n \in \tau, then\ \bigcap_{i=1}^{n} A_i \in \tau, For\ at\ least\ n\ elements\ B_1, B_2, .., B_n \in \tau, then\ \bigcap_{i=1}^{n} B_i \in \tau$
2) , ∩ and for at least n elements , ; [∩ ∩)]. Where n is finite. iii) For at least n elements , ∪ and for at least n elements , ; [∪ ∪)].

Proposition 2.4. Suppose that (X,τ) be an anti-topological space. Then:

1) $(X, \tau \cup \emptyset)$ is a neutro-topological space.
2) $(X, \tau \cup X)$ is a neutro-topological space

Remarks 1.4 : From Proposition 2.4, we obtain that a neutro-topological space can be obtained from every anti-topological space.
Example 2.8. Consider example 2.2 $(X, \tau \cup \emptyset) and\ (X, \tau \cup X) are$ neutro-topological space.
Proposition 2.5. Suppose that (X,τ) is a classical topological space. Then:
1- $(X, \tau - \emptyset)$ is a neutro – topological space.
2- $(X, \tau - X)$ is a neutro – topological space.

3 Rough Anti-topology

Definition 3.1. The lower rough anti-approximation $\underline{\sim A} = int(A^{\circ}) = Anti(int(A)$ (anti-interior) is the largest anti-open set contained in$A$. $\underline{\sim A} = int(A^{\circ}) = \bigcup\{O \in \tau: O \subseteq A\}$
The upper rough anti-approximation $\overline{\sim A} = int(A^c) = int(Cl(A)$ (anti-closure) of A is the smallest anti-closed set containing$A$.$\overline{\sim A} = \bigcap\{F: F = O^c, O \in \tau, A \subseteq F\} -$.
The Boundary is BND$(A) = \overline{\sim A} - \underline{\sim A}$. If BND$(A) \neq \emptyset$, we called A is rough.
Example 3.1 Consider example 2.2. Suppose that$A = \{b,c,d\}$. Now $\underline{\sim A} = = \{b,c\}$.and $\overline{\sim A} = int(A) = \{b,c,d\} = A$. Now, BND$(A) = \overline{\sim A} - \underline{\sim A} = \{d\}$. So, A is rough.

Proposition 3.1. Suppose that $\kappa = (X, R, K_\tau)$ is a topologized approximation space. If A and B are two subsets of X , then

1) $\overline{\sim A} \subseteq A \subseteq \underline{\sim A}$;
2) $\overline{\sim \emptyset} = \underline{\sim \emptyset}$ and $\overline{\sim X} = \underline{\sim X} = X$;
3) $\overline{\sim (A \cup B)} = \overline{\sim A} \cup \overline{\sim B}$;
4) $\underline{\sim (A \cap B)} = \underline{\sim A} \cap \underline{\sim A}$;
5) $If\ A \subseteq B, then\ \overline{\sim A} \subseteq \overline{\sim B}$;
6) $\underline{(A \cup B)} \supseteq \underline{A} \cup \underline{B}$;
7) $\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$;
8) $\overline{(A^c)} = \underline{(A)}^c$;
9) $\underline{(A^c)} = \overline{(A)}^c$.

Note that:

The difference between classical rough approximations and rough approximations using anti-topology

We can summarize the difference between classical rough approximations and rough approximations using anti-topology as follows:

Example 3.2. Suppose that $X = \{1,2,3,4,5\}$. Let A={2,3,4};

	Classical rough approximations	Rough approximations using anti-topology
Lower approximations A	{2,3}	{2}
Upper approximations A	{2,3,4}	{1,2,3,4}
B(A)	{4}	{1.3,4}

We note that, in the classical case, the boundaries are more precise with a lower ambiguity ratio.
But in the anti-topological case, the boundaries become wider with increased ambiguity, reflecting a more accurate model of uncertainty and approximation in complex environments.
Anti-topology allows representing more extensive cases of indeterminate membership, which is useful in applications requiring modeling of imprecision and fuzziness.

Conclusion

We introduce the concepts of rough upper and lower approximations on anti-topology. Some properties of approximations and these algebraic structures are studies. Moreover, we note that in the anti-topological case, the boundaries become wider with increased ambiguity, reflecting a more accurate model of uncertainty and approximation in complex environments. In addition, Anti-topology allows representing more extensive cases of indeterminate membership, which is useful in applications requiring modeling of imprecision and fuzziness.

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References

[1] Z. Pawlak, Rough sets, Int. J. Inf. Comp. Sci. 11 (1982) 341–356.
[2] Z. Bonikowaski, Algebraic structures of rough sets, in: W.P. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer-Verlag, Berlin, 1995, pp. 242–247.
[3] T. Iwinski, Algebraic approach to rough sets, Bull. Polish Acad. Sci. Math. 35 (1987) 673–683.
[4] J. Pomykala, J. A. Pomykala, The stone algebra of rough sets, Bull. Polish Acad. Sci. Math., 36 (1998), 495–508.
[5] Anitha K and Venkatesan P,Rough Sets on Topological Spaces,International Journal of Mathematics Trends and Technology – Volume 10 Number 1 – Jun 2014.
[6] M. S¸ahin, A. Kargın, M. Y¨ucel, Neutro-Topological Space and Anti-Topological Space, in: NeutroAlgebra Theory Volume I, The Educational Publisher Inc., 2021.
[7] Abd El-Monsef, M. E. , Kozae, A. M. and Iqelan, M. J.(2010) Near approximations in topological spaces, International Journal of Mathem
[8] Lashin, E.F., Kozae, A.M., Abokhadra, A. and Medhat, T. (2005), Rough set theory for topological spaces, International Journal of Approximate Reasoning 40: pp. 35-43.
[9] Qays R. Shakir, On Some Operators in Anti-Topological Spaces,Journal of Kufa for Mathematics and Computer Vol.11, No.2, Aug., 2024, pp 26-30 .