Journal of Insights in Basic and Applied Sciences
مجلة رؤى في العلوم الأساسية والتطبيقية
Volume 2, Issue 1, 2026, Pages: 192-199
Journal homepage: https://jibas.com.ly/index.php/jibas/en

Order And multiplication Reversing In Lattice Ordered Groups

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عكس الترتيب والضرب في المجموعات المرتبة شبكياً

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Received: November 30, 2025	Accepted: January 30, 2026	Published: February 13, 2026

Abstract:
For any lattice ordered group $\mathcal{G}$ there are two closely associated lattice ordered groups: $\mathcal{G}^R$, obtained from $\mathcal{G}$ by reversing the order, and $\mathcal{G}_W$, obtained from $\mathcal{G}$ by reversing the multiplication. The primary objective of this study is to demonstrate that, for any lattice ordered group, $\mathcal{G}^R$ and $\mathcal{G}_W$, are isomorphic. Furthermore, we investigate the relationships between $\mathcal{G}$, $\mathcal{G}^R$ and $\mathcal{G}_W$, and identify the identities that hold within each. Additionally, for any variety $\mathcal{V}$ of $\mathscr{L}$-group, we establish that the mapping $\theta: \mathcal{V} \rightarrow \mathcal{V}^R$ where $\mathcal{V}^R = \{\mathcal{G}^R : \mathcal{G} \in \mathcal{V}\}$, is both a lattice and a semigroup automorphism of the set of varieties of lattice ordered groups.

Keywords: Order, Revers, Lattice, Automorphism, Varieties.

الملخص
لكل زمرة مرتبة شبكياً (Lattice-ordered group) $\mathcal{G}$، توجد زمرتان مرتبتان شبكياً مرتبطتان بها ارتباطاً وثيقاً: $\mathcal{G}^R$ الزمرة التي يتم الحصول عليها عن طريق عكس الترتيب، والزمرة $\mathcal{G}_W$ التي يتم الحصول عليها عن طريق عكس عملية الضرب. تتمثل الأهداف الرئيسية لهذه الدراسة في إثبات أن الزمرتين $\mathcal{G}^R$ و $\mathcal{G}_W$ هما زمرتان متماثلتان شبكياً (Isomorphic lattice-ordered groups) لأي زمرة مرتبة شبكياً. كما نتناول دراسة العلاقات بين ($\mathcal{G}, \mathcal{G}^R$، $\mathcal{G}_W$)، المتطابقات التي تتحقق في كل منها. بالإضافة إلى ذلك، نثبت لأي تنوع (Variety) $\mathcal{V}$ من الزمر المرتبة شبكياً، أن التطبيق (Mapping) $\theta: \mathcal{V} \rightarrow \mathcal{V}^R$ حيث $\mathcal{V}^R = \{\mathcal{G}^R : \mathcal{G} \in \mathcal{V}\}$ يمثل تبياناً ذاتياً (Automorphism) لكل من الشبكية ونصف الزمرة (Semigroup) لمجموعة تنوعات الزمر المرتبة شبكياً.

الكلمات المفتاحية: الترتيب، الانعكاس، الشبكة، التماثل الذاتي، الأصناف.

Introduction

Recall that a lattice ordered group is a group endowed with a lattice structure that is compatible with the group operations:

$$a\left(x \bigvee y\right) b = (axb) \bigvee (ayb),$$

$$a(x \bigwedge y)b = (axb) \bigwedge (ayb)$$

In particular, by an $\mathscr{L}$-homomorphism (respectively, $\mathscr{L}$-isomorphism) between two $\mathscr{L}$-group is meant a mapping which is both a group and a lattice homomorphism (respectively, group and lattice isomorphism). Also, an $\mathscr{L}$-subgroup means a subgroup which is also a sublattice and an $\mathscr{L}$-ideal means a convex normal $\mathscr{L}$-subgroup. For any element x of an $\mathscr{L}$-group, we write

$$x^+ = x \vee 1, \quad x^- = x \wedge 1, \quad |x| = x^+ \vee x^-,$$

where 1 denotes the identity of $\mathcal{G}$.

Varieties of lattice-ordered groups ($\mathscr{L}$-groups) are classes of $\mathscr{L}$-groups defined by a set of algebraic **identities** . Below are the primary examples, ordered from smallest to largest in the lattice of varieties :

1. **Trivial Variety E:** Consists only of the single-element group $\{e\}$ and is defined by the identity $x = e$
2. **Abelian Variety $\mathcal{A}$:** The smallest non-trivial variety, defined by the commutativity identity $xy = yx$
3. **Representable Variety $\mathscr{R}$:** Contains all $\mathscr{L}$-groups that can be expressed as a subdirect product of **totally ordered groups** .It is defined by the identity

$$(x - 1(y \vee e)x) \wedge (y - 1 \vee e) = e$$

4. **Normal Valued Variety $\mathcal{N}$:** This is the largest proper variety of $\mathscr{L}$-groups. It is defined by the Wolfenstein identity :

$$(x \vee e)(y \vee e) \leq (y \vee e)2(x \vee e)2$$

5. **Scrimger Varieties) $\mathcal{S}n$:** A sequence of varieties that sit between the Abelian and non-Abelian varieties, often defined through specific **wreath products** of groups.
6. **The Variety of All $\mathscr{L}$-groups) $\mathscr{L}\ \mathcal{G}$:** The largest possible variety, containing every lattice-ordered group without additional identity constraints .

Notation1.1. For any $\mathscr{L}$-group $\mathcal{G}$, we denote by $\mathcal{V}(\mathcal{G})$ the variety of $\mathscr{L}$-groups generated by $\mathcal{G}$. We denote by $\mathcal{L}$ the lattice of varieties of $\mathscr{L}$-group.

We shall require the following useful observation.

Lemma1.2. (*Martinez, J.1974, p265-284*) If $\mathcal{U}$ and $\mathcal{V}$. are $\mathscr{L}$-group varieties, an $\mathscr{L}$-group $\mathcal{G}$ belongs to $\mathcal{U} \vee \mathcal{V}$ if and only if there exist $\mathscr{L}$-ideals $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{G}$ such that $\mathcal{G} / \mathcal{M} \in \mathcal{U}, \mathcal{G} / \mathcal{N} \in \mathcal{V}$ and $\mathcal{M} \cap \mathcal{N} = \{1\}$.

SECTION 2: DUALITY AND ORDER INVERSION.

Definition 2.1. For any $\mathscr{L}$-group Defining $\mathcal{G}^R$ and $\mathcal{G}_W$ In $\mathscr{L}$ -group theory, these are ways to transform a group $(\mathcal{G},\cdot,\leq)$:

G^R (The Opposite l-group): $\mathcal{G}^R = (\mathcal{G},\leq^R)$ denote the $\mathscr{L}$-group obtained from $\mathcal{G}$ by reversing the order; thus $a \leq^R b$ if and only if $b \leq a$ .keeping the same group operation but **reversing the lattice order**.

G^W (The Dual l-group): $\mathcal{G}_W$=($\mathcal{G}_W$, $\leq$) denote the $\mathscr{L}$-group obtained from $\mathcal{G}$ by reversing the multiplication. Specifically, for any,$a, b \in \mathcal{G}$,the new operation $*$ is defined by $a * b = b\, a$. **reversing the multiplication** while keeping the original lattice order.

That $\mathcal{G}^R$ and $\mathcal{G}$, are both $\mathscr{L}$-groups are easily verified. A variety $\mathcal{V}$ of $\mathscr{L}$-groups is reversible if $\mathcal{V} = \mathcal{V}^R$, where $\mathcal{V}^R = \{\mathcal{G}^R : \mathcal{G} \in \mathcal{V}\}$.

Example: A primary example of reversible varieties of lattice-ordered groups ($\mathscr{L}$-groups) obtained by reversing the order is the variety of **abelian $\mathscr{L}$-groups** ($\mathcal{A}$). For any abelian $\mathscr{L}$-group G, the $\mathscr{L}$-group $\mathcal{G}^R$ (same group operation, opposite order $a \leq^R b \Leftrightarrow b \leq a$) is also abelian and satisfies the same identities, making $\mathcal{A} = \mathcal{A}^R$ a reversible variety.

For any $\mathscr{L}$-group $(G, \leq)$, the reversed structure $(G, \geq)$ remains an $\mathscr{L}$-group, but the lattice operations are swapped:

New Join: $x \vee * \, y = x \wedge y$ (The old "meet" becomes the new "join").

New Meet: $x \wedge * \, y = x \vee y$ (The old "join" becomes the new "meet").

Examples:

1-Standard Integers ($\mathbb{Z}$): *Original:* $5 \vee 10 = 10$.

Reversed: $5 \leq^{*} 10$ actually means $10 \leq 5$ in the old system. Therefore, the "larger" number in the reversed order is 5.

Calculation: $5 \vee^{*} 10 = min\ (5,10) = 5$.

2-Product Group ($\mathbb{Z} \times \mathbb{Z}$): *Original:* $(1,5) \vee (2,3) = (2,5)$.

Reversed: We take the component-wise minimum instead of the maximum.

Calculation: $(1,5) \vee^{*} (2,3) = (min\ (1,2), min\ (5,3)) = (1,3)$.

3-Lexicographic Product ($\mathbb{Z} \times \vec{\mathbb{Z}}$): *Original:* $(1,-100) > (0,50)$.

Reversed: the "positive" cone is flipped. Now, $(a,b) >^{*} (0,0)\ if\ a < 0, or\ if\ a = 0\ and\ b < 0$.

Calculation: $(0,50)$ is now "larger" than $(1,-100)$.

Definition2.2: A variety $\mathcal{V}$ of $\mathscr{L}$-groups is reversible if $\mathcal{V}$=$\mathcal{V}^{R}$, where $\mathcal{V}^{R}$=$\{\ \mathcal{G}^{\,R}\!: \ \mathcal{G} \in V\}$.

Other Examples: While $\mathcal{A}$ is the most straightforward example, many varieties generated by particular types of $\mathscr{L}$-groups are not reversible, as $\mathcal{G}^{R}$ is not always isomorphic to $\mathcal{G}$ in terms of lattice structure, even if they share the same algebraic group structure.

The reversing of order turns meets into joins ($\vee \rightarrow \wedge$), which is why only specific structures (like abelian or certain other special classes) are reversible.

Notation 2.3. For any $\mathscr{L}$-group $\mathcal{G}$, we denote the lattice operations in $\mathcal{G}^{R}$ by $\vee^{R} \wedge^{R}$ and write, for $x \in \mathcal{G}$,

$$x^{+R} = x \vee^{R} 1, \qquad x^{-R} = (x \wedge^{R} 1)^{-1}, \qquad |x|^{R} = x^{+R} \vee^{R} x^{-R}.$$

Note that for x, y in $\mathcal{G}$,

$$x \vee^{R} y = x \wedge y$$

and

$$x \wedge^{R} y = x \vee y.$$

Lemma 2.4. (Reilly, N. R. Huss, E. M., 1984, 176-191) The mapping $\varphi\colon g \rightarrow g^{-1}$ defined by $\varphi(g) = g^{-1}$ is an $\mathscr{L}$-isomorphism of $\mathcal{G}^{R}$ onto $\mathcal{G}_{\mathcal{W}}$.

Proof: For $g, h \in \mathcal{G}^{R}$,

$$\varphi(gh) = (gh)^{-1} = h^{-1}g^{-1} = \varphi(g)\varphi(h)$$

so that φ is a group isomorphism. Since

$$\varphi\left(g \bigvee^{\mathcal{R}} h\right) = \left(g \bigvee^{\mathcal{R}} h\right)^{-1}$$

$$= (g \bigwedge h)^{-1}$$

$$= g^{-1} \bigvee h^{-1}$$

$$= \varphi(g) \bigvee \varphi(h)$$

and similarly,

$$\varphi(g \bigwedge^{R} h) = \varphi(g) \bigwedge \varphi(h)$$

we see that φ, is also a lattice isomorphism.

Corollary 2.5. (Huss, E. M., 1984) For any $\mathscr{L}$-group $\mathcal{G}$, $\mathcal{V}(\mathcal{G}^R) = \mathcal{V}(\mathcal{G}_W)$.

In the light of Lemma 2.4, we may work with whichever is the more convenient of $\mathcal{G}^R$ or $\mathcal{G}_W$, in any given situation.

Although, as we shall see, some of the properties of $\mathcal{G}$ and $\mathcal{G}^R$ can be quite different, some features are invariant under order reversal.

Lemma 2.6. Let $\mathcal{G}$ be an $\mathscr{L}$-group and $\mathcal{H} \subseteq \mathcal{G}$. Then $\mathcal{H}$ is a sublattice (respectively, subgroup, $\mathscr{L}$-subgroup or $\mathscr{L}$-ideal) of $\mathcal{G}$ if and only if $\mathcal{H}^{\mathcal{R}}$ is a sublattice (respectively subgroup, $\mathscr{L}$-subgroup or $\mathscr{L}$-ideal) of $\mathcal{G}^R$.

Furthermore, if $\mathcal{H}$ is an $\mathscr{L}$-ideal, then $(\mathcal{G}/\mathcal{H})^{\mathrm{R}}$ is $\mathscr{L}$-isomorphic to $\mathcal{G}^R/\mathcal{H}^{\mathcal{R}}$.

The next observation follows easily from the fact that the lattice structure of an $\mathscr{L}$-group is distributive.

Lemma 2.7. (Huss, E. M., 1984) Let $\mathcal{F}$ be the free $\mathscr{L}$-group on the non-empty set $\mathcal{X}$. Then any element $u \in \mathcal{F}$ can be written in the form

$$u = \bigvee_{I} \bigwedge_{J} \prod_{K} x_{ijk}$$

where I, J and K are finite sets and $x_{ijk} \in \mathcal{X} \cup \mathcal{X}^{-1} \cup \{1\}$ for all $i \in I, j \in J, k \in K$.

Lemma 2.8. (Huss, E. M., 1984) For any $\mathscr{L}$-group $\mathcal{G}$ and any $u = \bigvee_I \bigwedge_J \prod_K x_{ijk} \in \mathcal{F}$ the following are equivalent.

(i) The identity $u = 1$ holds in $\mathcal{G}$.

(ii) The identity $u^R = 1$ holds in $\mathcal{G}^R$.

(iii) The identity $u^1 = 1$ holds in $\mathcal{G}^R$.

Proof, clearly $\mathcal{G}$ satisfies the law $\bigvee_I \bigwedge_J \prod_K x_{ijk} = 1$ if and only if $\mathcal{G}^R$ satisfies the law $\bigvee_i^R \bigwedge_j^R \prod_k x_{ijk} = 1$. But

$$\bigvee_{i}^{R} \bigwedge_{j}^{R} \prod_{k} x_{ijk} = 1$$

$$\Leftrightarrow \bigvee_{I}^{R} \left(\bigwedge_{J}^{R} \left(\prod_{K} x_{ijk} \right)^{-1} \right)^{-1} = 1$$

$$\Leftrightarrow \left(\bigvee_{I}^{R} \bigwedge_{J}^{R} \left(\prod_{K} x_{ijk} \right)^{-1} \right)^{-1} = 1$$

$$\Leftrightarrow \bigvee_{I}^{R} \bigwedge_{J}^{R} \left(\prod_{K} x_{ijk} \right)^{-1} = 1$$

Thus $\mathcal{G}$ satisfies a law $u = 1$ if and only if $\mathcal{G}^R$ satisfies $u^R = 1$, which establishes the equivalence of (i) and (ii).

Now the mapping $x \longrightarrow x^{-1}$ $(x \in X)$ of $\mathcal{X}$ into $\mathcal{F}$ clearly extends to an automorphism φ, say, of $\mathcal{F}$. Hence the identity holds in $\mathcal{G}^R$ if and only if the identity $u^R\ \varphi = 1$ holds in $\mathcal{G}^R$. Since

$$u^R\ \varphi = \bigvee_I^R \bigwedge_J^R \left(\prod_K x_{ijk}\right)^{-1} \varphi$$

$$= \bigvee_I^R \bigwedge_J^R \prod_K (x_{ijk})^{-1}\ \varphi$$

$$= \bigvee_I^R \bigwedge_J^R \prod_K (x_{ijk}\varphi)^{-1}$$

$$= \bigvee_i^R \bigwedge_j^R \prod_k x_{ijk}$$

the equivalence of (ii) and (iii) follows.

Corollary 2.9. (Huss, E. M., 1984) For any variety of $\mathscr{L}$-group $\mathcal{V}$, $\mathcal{V}^R$ is a variety. Moreover, the following are equivalent.

(i) $\mathcal{V}$ has a basis of identities $[u_\alpha = 1, \alpha \in \mathcal{A}]$.

(ii) $\mathcal{V}^R$ has a basis of identities $[u_\alpha^{\mathcal{R}} = 1, \alpha \in \mathcal{A}]$.

(iii) $\mathcal{V}^R$ has a basis of identities $[u_\alpha^1 = 1, \alpha \in \mathcal{A}]$.

This leads naturally to the question of whether or not it is always the case that, $\mathcal{V} = \mathcal{V}^R$ or, equivalently, whether or not it is the case that for all $\mathscr{L}$-groups $\mathcal{G}$ the varieties $\mathcal{V}\ (\mathcal{G}^R)$ and $\mathcal{V}\ (\mathcal{G}^{\mathrm{R}}) = \mathcal{V}\ (\mathcal{G}_W)$ are always the same.

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3. AN AUTOMORPHISM OF $\mathscr{L}$

Since there exist $\mathcal{V} \in \mathscr{L}$ such that $\mathcal{V} \neq \mathcal{V}^R$, we consider the basic properties of the mapping $\mathcal{V} \to \mathcal{V}^R$ in this section.

Notation 3.1. Let $\theta: \mathcal{V} \to \mathcal{V}^R$ be the mapping defined by

$$\mathcal{V}\ \theta = \mathcal{V}^R \quad (\mathcal{V} \in \mathscr{L}),$$

and let

$$\mathcal{F} = \{\mathcal{V} \in \mathcal{L} : \mathcal{V}^{\mathrm{R}} = \mathcal{V}\}.$$

Theorem 3.2. The mapping θ is a lattice automorphism of $\mathscr{L}$ with the following properties:

(i) θ^2 is the identity mapping;

(ii) θ preserves arbitrary joins and meets;

(iii) $\mathcal{F}$ is a complete sublattice of $\mathscr{L}$;

(iv) for any $\mathcal{V} \in \mathscr{L}$, $\mathcal{V} \vee \mathcal{V}^R \in \mathcal{F}$ and $\mathcal{V} \wedge \mathcal{V}^R \in \mathcal{F}$.

Proof: For any word $u \in \mathcal{F}$, it is clear that $(u^R)^R$so that by Corollary 2.8, we have$\mathcal{V}\theta^2 = \mathcal{V}$, for all $\mathcal{V} \in \mathscr{L}$. Thus (i) holds. In addition, for any $\mathscr{L}$-group G,

$$\mathcal{G} \in \mathcal{V}\ \theta \Leftrightarrow \mathcal{G}^R \in \mathcal{V}\theta^2 = \mathcal{V}.$$

Hence, for any family $\{\mathcal{V}_\alpha : \alpha \in \mathcal{A}\} \subseteq \mathscr{L}$,

$$\mathcal{G} \in \left(\bigwedge \mathcal{V}_\alpha\right)\theta \Leftrightarrow \mathcal{G}^{\mathcal{R}} \in \bigwedge \mathcal{V}_\alpha$$

$$\mathcal{G}^{\mathcal{R}} \in \mathcal{V}_\alpha, \quad \text{for all } \alpha \in \mathcal{A},$$

$$\mathcal{G}^{\mathcal{R}} \in \mathcal{V}_\alpha^{\mathcal{R}}, \quad \text{for all } \alpha \in \mathcal{A},$$

$$\mathcal{G} \in \bigwedge \mathcal{V}_\alpha^{\mathcal{R}} = \bigwedge {\mathcal{V}_\alpha}_\alpha \theta$$

and θ respects arbitrary meets.

Now suppose that $\mathcal{G} \in (\vee \mathcal{V}_\alpha)\theta$. Then $\mathcal{G}^{\mathcal{R}} \in \vee \mathcal{V}_\alpha$ and there exist$V_\alpha \in \mathcal{V}(\alpha \in \mathcal{A})$, a subdirect product $\mathcal{H}$ of the V_α, and an $\mathscr{L}$-epimorphism $\psi: \mathcal{H} \to \mathcal{G}^{R}$:

$$\mathcal{H} \subseteq \prod V_\alpha$$

Clearly the same mapping ψ gives an $\mathscr{L}$-epimorpbism $\psi: \mathcal{H}^{\mathcal{R}} \to \mathcal{G}^{\mathcal{R}^{\mathcal{R}}}$ where $\mathcal{H}^{\mathcal{R}} \subseteq \prod V_\alpha$ so that $\mathcal{G} \in \vee \mathcal{V}_\alpha^{\mathcal{R}}$).
Thus

$$\left(\bigvee (\mathcal{V}_\alpha)\theta\right) \subseteq \bigvee \mathcal{V}_\alpha^{\mathcal{R}} = \bigvee \mathcal{V}_\alpha \theta$$

By applying the inverse map and following the same logical steps in reverse order, we have the reverse containment and therefore

$$\left(\bigvee (\mathcal{V}_\alpha)\theta\right) = \bigvee \mathcal{V}_\alpha \theta$$

and θ preserves arbitrary joins. This establishes (ii). Since (i) clearly implies that θ is a bijectioc, it now follows that θ is an automorphism.

Property (iii) follows immediately from (ii).

(iv) For any $\mathcal{V} \in \mathscr{L}$ we have

$$\left(\mathcal{V} \bigvee \mathcal{V}^{\mathcal{R}}\right)\theta = \left(\mathcal{V} \bigvee \mathcal{V}\theta\right)\theta$$

$$= \left(\mathcal{V}\theta \bigvee \mathcal{V}\theta^2\right)$$

$$= \left(\mathcal{V}\theta \bigvee \mathcal{V}\right)$$

$$= \left(\mathcal{V} \bigvee \mathcal{V}^{\mathcal{R}}\right)$$

by (i) and since θ is an automorphism. Hence

$$\mathcal{V} \bigvee \mathcal{V}^{\mathcal{R}} \in \mathcal{F}$$

and similarly

$$\mathcal{V} \bigwedge \mathcal{V}^{\mathcal{R}} \in \mathcal{F}$$

As an immediate consequence of Theorem 3.2(iii) we have the following corollary (Aboujanah, A., et al, 2025).

Corollary 3.3. Let $\mathcal{V} \in \mathscr{L}$. Then the following are equivalent:

(i) $\mathcal{V} \in \mathcal{F}$;

(ii) $\mathcal{V}$ and $\mathcal{V}^R$ are comparable.

We now consider how θ behaves relative to the semigroup structure of $\mathscr{L}$.

Definition 3.4. For $\mathcal{U}, \mathcal{V} \in \mathscr{L}$, we denote by $\mathcal{UV}$ the class of all $\mathscr{L}$-groups G for which there exists an $\mathscr{L}$-ideal H with $H \in \mathcal{U}$ and $G/H \in \mathcal{V}$ and refer to $\mathcal{UV}$ as the product of the varieties $\mathcal{U}$ and $\mathcal{V}$. A variety $\mathcal{W}$ is said to be indecomposable if $\mathcal{W} = \mathcal{UV}$ implies that either $\mathcal{U}$ or $\mathcal{V}$ is the trivial variety.

It was observed by (Martinez, J. 1972, 535-553) that $\mathscr{L}$ is a semigroup with respect to the above defined product of varieties.

Definition 3.5. Let G be an $\mathscr{L}$-group. If, for every $g \in G$ and every convex $\mathscr{L}$-subgroup M that is maximal with respect to not containing g, M is normal in the convex $\mathscr{L}$-subgroup generated by M and g, then G is said to be normal valued. The class of all normal valued $\mathscr{L}$-groups will be denoted by $\mathcal{N}$.

Lemma 3.6 (Bigard, A., Keimel, K., & Wolfenstein, S. 1977) The class $\mathcal{N}$ is the variety of $\mathscr{L}$-group defined by the identity

$$(x \vee 1)(y \vee 1) \leq (y \vee 1)^2 (x \vee 1)^2$$

And is the largest varieties of $\mathscr{L}$-group.

Theorem 3.7 (Glass, A. M. W., Holland, W. C., & McCleary, S. H. 1980, 1-20). The set $\mathscr{L}$ of proper varieties of $\mathscr{L}$-groups other than $\mathcal{N}$ forms a free semigroup on the set of indecomposable varieties.

Theorem 3.8. The mapping θ is an automorphism of the semigroup structure of $\mathcal{L}$.

Proof: Since θ is bijective, it remains to show that θ is a semigroup

homomorphism. Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}$. Then, by Lemma 2.6,

$$G \in (\mathcal{UV})\theta \Leftrightarrow G^R \in \mathcal{UV}$$

$\Leftrightarrow$ there exists an $\mathscr{L}$-ideal H of G^R with $H \in \mathcal{U}$ and $G^R/H \in \mathcal{V}$

$\Leftrightarrow$ there exists an $\mathscr{L}$-ideal K of G ($K = H^R$) with $K \in \mathcal{U}\theta$ and G/K ($\cong (G^R/H)^R$) $\in \mathcal{V}\theta$

$\Leftrightarrow G \in (\mathcal{U}\theta)(\mathcal{V}\theta)$

Thus $(\mathcal{UV})\theta = (\mathcal{U}\theta)(\mathcal{V}\theta)$, as required.

Theorem 3.8 together with the next observation will enable us to make some observations regarding $\mathcal{F}$ as a subset of the semigroup $\mathcal{L}$.

Proposition 3.9. Let $\mathcal{U} \in \mathcal{L}^{*}$ and let $\mathcal{U} = \mathcal{U}_1 \ldots\ldots \mathcal{U}_n$ where each $\mathcal{U}_i \in \mathcal{L}$ $(i = 1, 2, 3, \ldots\ldots, n)$ is indecomposable. Then $\mathcal{U} \in \mathcal{F} \Leftrightarrow \mathcal{U}_i \in \mathcal{F}$, for all i.

Proof: From Theorem 3.8, we have

$$\mathcal{U}\theta = (\mathcal{U}_1, \ldots\ldots, \mathcal{U}_n)\theta$$

$$= \mathcal{U}_1\theta, \ldots\ldots, \mathcal{U}_n\theta$$

where, since θ is an automorphism of the semigroup $\mathcal{L}$, each $\mathcal{U}_i\theta$ (i = l,..., n) must be indecomposable. Since the factorization of varieties in $\mathcal{L}^*$ into indecomposable varieties is unique, by Theorem 3.7 it follows that

Thus

$$\mathcal{U} \in \mathcal{F} \Leftrightarrow \mathcal{U}_i \in \mathcal{F} \; for \; i = 1, 2, 3, \ldots\ldots, n$$

Corollary 3.10. The complement $\mathcal{F}^c$ of $\mathcal{F}$ in $\mathscr{L}$ is a prime semigroup ideal (that is, $\mathcal{UV} \in \mathcal{F}^c$ *implies* $\mathcal{U} \in \mathcal{F}^c$ *or* $\mathcal{V} \in \mathcal{F}^c$). In particular, $\mathcal{F}$ and $\mathcal{F}^c$ are both sub semigroups of $\mathscr{L}$.

Proof: By their distinguished positions in the lattice $\mathcal{L}$ it is clear that the trivial variety, the variety of all $\mathscr{L}$-group, and the variety,$\mathcal{N}$ all lie in $\mathcal{F}$.

Thus $\mathcal{F}^{c} \subseteq \mathcal{L}^*$. The result now follows from theorem 3.8 and proposition 3.9.

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