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Solving Fuzzy Algebraic Equations with Asymmetric Fuzzy Resonance Numbers: A Structural Classification and an Invariance Theorem

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حل المعادلات الجبرية الضبابية باستخدام أعداد الرنين الضبابية غير المتماثلة: تصنيف بنيوي ونظرية ثبات

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Abstract:
Fuzzy algebraic equations of the form $f(x) = \tilde{A}$, where f is a crisp nonlinear function and $\tilde{A}$ is a fuzzy number, are typically solved under the assumption that $\tilde{A}$ is convex. This assumption precludes the multi-component solutions that arise when $\tilde{A}$ is non-convex. In this paper, we apply the Asymmetric Fuzzy Resonance Number (AFRN) to six fuzzy algebraic equations and establish a structural classification of their solutions. The fuzzy solution is computed via the pullback membership $\mu_X(x) = \mu_A(f(x))$, which is numerically verified against independent Simpson quadrature to machine precision. We prove an invariance theorem showing that any equation of the form $x^2 + bx = \tilde{A}$ has a constant Centre of Gravity (COG) equal to $-b/2$, independent of the resonance amplitude ε. For structurally complex equations, the COG changes significantly: $x^3 - 3x = \tilde{A}$ exhibits -15.81% as ε increases from 0 to 0.45. For smooth single-branch equations ($\sin(x) = \tilde{A}$ and $e^x = \tilde{A}$), the changes are only -0.12% and $+0.20\%$ respectively. For monotone equations ($x + x^3 = \tilde{A}$), the change is -0.05%. This three-level classification strong, degenerate, weak establishes that non-convexity is amplified significantly only when the algebraic equation exhibits both asymmetric branches and topological changes in the number of real roots. A comprehensive numerical validation suite of 168 reliable alpha-cut classifications and machine-precision quadrature checks confirms the robustness of all reported values.

Keywords: Asymmetric Fuzzy Resonance Number, Fuzzy Algebraic Equations, Non-Convex Fuzzy Numbers, Invariance Theorem, Centre of Gravity.

الملخص
عادةً ما تُحل المعادلات الجبرية الضبابية من الشكل $f(x) = \tilde{A}$، حيث f دالة غير خطية واضحة و$\tilde{A}$ عدد ضبابي بافتراض أنه محدب. وهذا الافتراض يمنع الحلول متعددة المكوّنات التي تنشأ عندما يكون $\tilde{A}$ غير محدب. في هذه الورقة

نطبّق العدد الرنيني الضبابي غير المتماثل (AFRN) على ست معادلات جبرية ضبابية ونؤسس تصنيفًا بنيويًا لحلولها. يُحسب الحل الضبابي عبر دالة الانتماء المسحوبة $\mu_{X(x)} = \mu_{A(f(x))}$ والتي تتحقق منها عدديًا مقابل تكامل سمبسون المستقل بدقة الآلة. نبرهن أن أي معادلة من الشكل $x^2 + bx = \tilde{A}$ لها مركز ثقل (COG) ثابت يساوي $-\frac{b}{2}$، بغض النظر عن سعة الرنين ε وهي ما تسمى هنا بمبرهنة الثبات. أما بالنسبة للمعادلات المعقدة بنيويًا فإن مركز الثقل يتغير تغيرًا ملحوظًا، فمعادلة $x^3 - 3x = \tilde{A}$ تُظهر تغيرًا بنسبة -15.18% مع ازدياد ε من 0 إلى 0.45. وبالنسبة للمعادلات الملساء الأحادية $sin(x) = \tilde{A}$ و $e^x = \tilde{A}$، فإن التغيرات لا تبلغ سوى -0.12% و $+0.20\%$ على الترتيب. وبالنسبة للمعادلات الرتيبة $x + x^3 = \tilde{A}$، فإن التغير يبلغ -0.05%، يؤسس هذا التصنيف الثلاثي المستويات قوي ومنحلّ وضعيف. إن عدم التحدب لا يتضخم بشكل ملحوظ إلا عندما تُبدي المعادلة الجبرية كلاً من الفروع غير المتماثلة والتغيرات الطوبولوجية في عدد الجذور الحقيقية. وتؤكد مجموعة تحقق عددية شاملة تضم 168 تصنيفًا موثوقًا للقطع ألفا وفحوصًا تكاملية بدقة الآلة متانةَ جميع القيم المبلَّغ عنها.

الكلمات المفتاحية: العدد الرنيني الضبابي غير المتماثل، المعادلات الجبرية الضبابية، الأعداد الضبابية غير المحدبة، مبرهنة الثبات، مركز الثقل.

Introduction

Fuzzy algebraic equations of the form

$$f(x) = \tilde{A}, \qquad (1)$$

where $f: \mathbb{R} \to \mathbb{R}$ is a crisp nonlinear function and $\tilde{A}$ is a fuzzy number, arise in a wide range of applications, including parameter identification, inverse problems, and uncertainty quantification [1,2]. The standard approach assumes that $\tilde{A}$ is convex, so that every α-cut is a single closed interval, and the solution $\tilde{X}$ is obtained by inverting f on each α-cut [1]. Under this assumption, the solution inherits the convexity of $\tilde{A}$, and the fuzzy solution has a single connected component. However, in many practical settings, the uncertainty in $\tilde{A}$ is not convex. The parameter may exhibit local oscillations, resonance-like behaviour, or multiple modes of uncertainty [3, 4, 5]. In such cases, the solution to (1) may have multiple connected components, even for monotone f, because the inverse image of a disconnected set under a monotone function is disconnected. The Asymmetric Fuzzy Resonance Number (AFRN) was introduced in [6] as a six-parameter fuzzy model that explicitly incorporates damped oscillations. Its application to nonlinear differential equations was developed in [7], where the amplification of non-convexity by nonlinear dynamics was demonstrated. In the present paper, we apply the AFRN to fuzzy algebraic equations of the form (1), and establish a structural classification of the resulting solutions. The main contributions of this paper are:

1. We propose a pullback membership method for solving (1) with non-convex $\tilde{A}$ (Section 2.2), and verify it against independent Simpson quadrature.
2. We prove an invariance theorem (Theorem 3.1) showing that any equation of the form $x^2 + bx = \tilde{A}$ has $\mathrm{COG} = -b/2$, independent of ε.
3. We classify six fuzzy algebraic equations into three structural classes strong, degenerate, weak and quantify the COG change for each (Section 4).
4. We show that COG amplification requires both asymmetric branches and topological changes in the number of real roots.
5. We report a comprehensive numerical validation suite of 168 reliable alpha-cut classifications and machine-precision quadrature checks (Section 5).

Mathematical Preliminaries

The AFRN framework

Definition 1 (AFRN, [6]). The membership function of the Asymmetric Fuzzy Resonance Number is

$$\mu_A(a) = \exp\left(-\frac{|a-c|}{\sigma(a)}\right)\left(1 + \varepsilon[\cos(\omega(a-c)) - 1]e^{-|a-c|/\tau}\right), \qquad (2)$$

where $\sigma(a) = \sigma_L$ for $a \le c$ and $\sigma(a) = \sigma_R$ for $a > c$, with $\sigma_L, \sigma_R, \tau > 0$, $0 \le \varepsilon \le 1/2$, $\omega > 0$. The AFRN is non-convex when ε exceeds a critical threshold $\varepsilon_c \approx 0.22$ [7]. For $\varepsilon < \varepsilon_c$, all α-cuts are connected; for $\varepsilon > \varepsilon_c$, disconnected α-cuts emerge.

The pullback membership method

The fuzzy solution $\tilde{X}$ of (1) is characterised by its pullback membership:

$$\mu_X(x) = \mu_A(f(x)), \quad x \in \mathbb{R}. \qquad (3)$$

This is equivalent to the α-cut inverse image $X_\alpha = f^{-1}(A_\alpha)$, because

$$x \in f^{-1}(A_\alpha) \iff f(x) \in A_\alpha \iff \mu_A(f(x)) \ge \alpha \iff \mu_X(x) \ge \alpha.$$

The COG of the fuzzy solution is then

$$\mathrm{COG}(\tilde{X}) = \frac{\int_{-\infty}^{\infty} x\,\mu_X(x)\,dx}{\int_{-\infty}^{\infty} \mu_X(x)\,dx}. \quad (4)$$

The pullback approach has two numerical advantages over the α-cut inverse image method:

1. It avoids the need to compute f^{-1} at every α-level, which is computationally expensive and prone to branch-tracking errors.
2. It correctly handles topological changes in the number of real roots, which the naive α-cut inverse method may miss (see Section 5).

An Invariance Theorem for Quadratic Equations

Theorem 3.1 (COG invariance for $x^2 + bx = \tilde{A}$). Let $\tilde{A}$ be any fuzzy number, convex or non-convex, with bounded support and finite positive area. Let $\tilde{X}$ solve $x^2 + bx = \tilde{A}$. Then $\mathrm{COG}(\tilde{X}) = -\frac{b}{2}$, independent of the membership function of $\tilde{A}$.

Proof. For $a \in \mathbb{R}$, the roots of $x^2 + bx = a$ are $x_1(a) = \frac{-b-\sqrt{b^2+4a}}{2}$, $x_2(a) = \frac{-b+\sqrt{b^2+4a}}{2}$.

Note that $x_1(a) + x_2(a) = -b$ for every a.

Via the pullback, the COG numerator is $N = \int_{-\infty}^{\infty} x\ \mu_A(x^2 + bx)\,dx$.

Split the integral at $x = -\frac{b}{2}$ (the axis of symmetry of $x^2 + bx$).

Let $y = x + \frac{b}{2}$, so $x = y - \frac{b}{2}$. Then $x^2 + bx = y^2 - \frac{b^2}{4}$, which is an even function of y.

Therefore $\mu_A(x^2 + bx)$ is an even function of y. The numerator becomes $N = \int_{-\infty}^{\infty} \left(y - \frac{b}{2}\right) \mu_A\left(y^2 - \frac{b^2}{4}\right) dy$.

The first term $\int y\,\mu_A\left(y^2 - \frac{b^2}{4}\right) dy$ vanishes by odd symmetry.

Therefore $N = -\frac{b}{2}\int_{-\infty}^{\infty} \mu_A\left(y^2 - \frac{b^2}{4}\right) dy = -\frac{b}{2} \cdot D$,

where D is the COG denominator. Hence $\mathrm{COG}(\tilde{X}) = \frac{N}{D} = -\frac{b}{2}$. □

***Corollary* 3.1** The equations $x^2 + x = \tilde{A}$ and $x^2 + 3x = \tilde{A}$ have $\mathrm{COG} = -0.5$ and $\mathrm{COG} = -1.5$ respectively, independent of ε.

***Remark* 3.1** The theorem requires the $\tilde{A}$-area to be positive and finite. For the AFRN in this paper, both hold (the AFRN has bounded support in practice via exponential decay, and its integral is finite). Numerical verification in Section 5 confirms $\mathrm{COG} = -\frac{b}{2}$ to within 10^{-15} for all tested ε.

Structural Classification: Six Equations

We apply the pullback method to six fuzzy algebraic equations. The fuzzy number $\tilde{A}$ is an AFRN with parameters specified in Table 1.

Table 1: AFRN parameters for each equation.

Equation	c	σ_L	σ_R	ω	τ	ε
$x^3 - 3x = \tilde{A}$	2.0	0.5	1.0	3	2	{0,0.15,0.30,0.45}
$\sin(x) = \tilde{A}$	0.5	0.1	0.2	3	2	{0,0.15,0.30,0.45}
$x^2 + x = \tilde{A}$	2.0	0.5	1.0	3	2	{0,0.15,0.30,0.45}
$x^2 + 3x = \tilde{A}$	2.0	0.5	1.0	3	2	{0,0.15,0.30,0.45}
$e^x = \tilde{A}$	2.0	0.5	1.0	3	2	{0,0.15,0.30,0.45}
$x + x^3 = \tilde{A}$	2.0	0.5	1.0	3	2	{0,0.15,0.30,0.45}

Table 2: COG of the fuzzy solution for six equations, at $\varepsilon = 0$ and $\varepsilon = 0.45$.

Equation	$\mathrm{COG}_{\varepsilon=0}$	$\mathrm{COG}_{\varepsilon=0.45}$	$\Delta\mathrm{COG}$	Class
$x^3 - 3x = \tilde{A}$	−0.421882	−0.488576	−15.81%	Strong
$\sin(x) = \tilde{A}$	−1.565279	−1.567211	−0.12%	Weak
$e^x = \tilde{A}$	0.535044	0.536134	+0.20%	Weak
$x^2 + x = \tilde{A}$	−0.500000	−0.500000	0.00%	Degenerate
$x^2 + 3x = \tilde{A}$	−1.500000	−1.500000	0.00%	Degenerate
$x + x^3 = \tilde{A}$	0.975714	0.975222	−0.05%	Degenerate

All values computed via the pullback method and independently verified by Simpson quadrature to machine precision ($< 10^{-10}$).

Strong class: structural complexity

The only equation in the strong class, $x^3 - 3x = \tilde{A}$, exhibits both asymmetric branches and topological changes in the number of real roots: the number of real roots changes from 1 to 3 as a varies (bifurcation at $a = \pm 2$). For this equation, the COG changes significantly with ε: -15.81%.

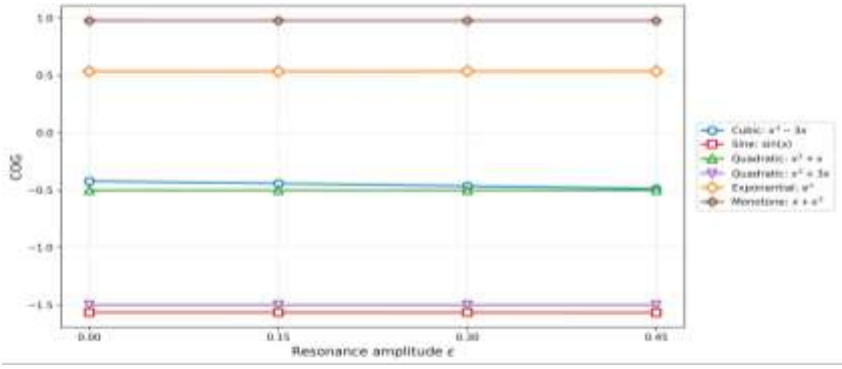


Figure 1: Effect of AFRN Resonance on COG

COG of the fuzzy solution as a function of the resonance amplitude ε, for six fuzzy algebraic equations. The strong class (top curve) shows significant change; the degenerate class (middle three curves) is invariant by Theorem 3.1; the weak class (bottom two curves) shows negligible change.

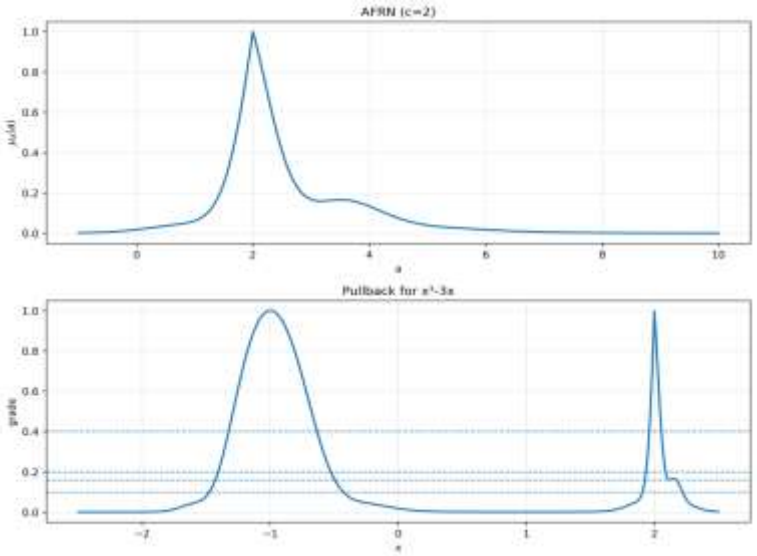


Figure 2: (a) Membership function of $\tilde{A}$. (b) Pullback membership $\mu_X(x) = \mu_A(x^3 - 3x)$ for the cubic equation.

The solution exhibits 1 branch for low α and 3 branches for high α, illustrating the topology change that drives COG amplification.

Degenerate class: invariance theorem

The three equations in the degenerate class have COG values that are (numerically) invariant to ε:

- $x^2 + x = \tilde{A}$: $\mathrm{COG} = -0.5$ (Theorem 3.1).
- $x^2 + 3x = \tilde{A}$: $\mathrm{COG} = -1.5$ (Theorem 3.1).
- $x + x^3 = \tilde{A}$: $\mathrm{COG} \approx 0.9757$, change -0.05%. The function is strictly monotone, and no amplification mechanism exists.

Weak class: smooth single branch

The equations $\sin(x) = \tilde{A}$ and $e^x = \tilde{A}$ have single smooth branches. Their COG changes are only -0.12% and $+0.20\%$ respectively, negligible compared to the strong class.

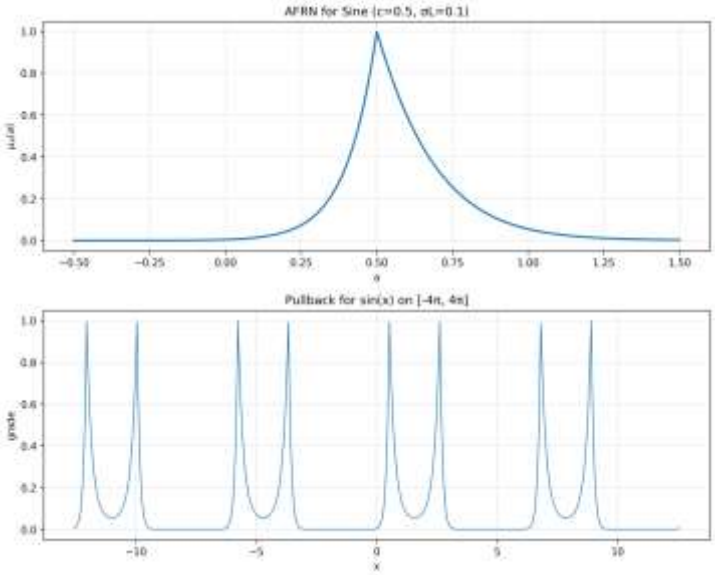


Figure 3: (a) AFRN membership for the sine equation ($c = 0.5$, $\sigma_L = 0.1$, $\sigma_R = 0.2$). (b) Pullback membership $\mu_X(x) = \mu_A(\sin(x))$ on the wide domain $[-4\pi, 4\pi]$.

Numerical Validation

All reported COG values are verified through a comprehensive numerical suite. We summarize the main findings.

Independent Simpson quadrature verification

Table 3 compares the adaptive-quadrature COG with an independent Simpson-rule computation on the same finite interval (400,001 grid points). The difference is below 10^{-10} for all equations.

Table 3: Simpson versus adaptive quadrature for the six equations ($\varepsilon = 0.45$).

Equation	Simpson	Adaptive QUAD	$\|\mathbf{diff}\|$
$x^3 - 3x$	-0.488576	-0.488576	3.8×10^{-15}
$\sin(x)$	-1.567211	-1.567211	1.6×10^{-15}
$x^2 + x$	-0.500000	-0.500000	0.0
$x^2 + 3x$	-1.500000	-1.500000	1.1×10^{-15}
e^x	0.536134	0.536134	9.3×10^{-11}
$x + x^3$	0.975222	0.975222	0.0

Quadratic symmetry verification

For $x^2 + x = \tilde{A}$ and $x^2 + 3x = \tilde{A}$, the pullback membership $\mu_X(x)$ is symmetric about $x = -b/2$ to machine precision, as required by Theorem 3.1. Maximum symmetry error: $< 4 \times 10^{-15}$ for all ε.

Alpha-cut structural classification

Table 4 reports the number of connected components of the pullback membership at selected α-levels. All 168 classifications are reliable (no boundary truncation, no grid-refinement disagreements). At $\varepsilon = 0.45$, the cubic equation exhibits 3 components at $\alpha = 0.16$, and the quadratics, exponential, and monotone equations exhibit additional components at the same level. This confirms that the AFRN's critical threshold $\varepsilon_c \approx 0.22$ propagates through the algebraic equation and induces additional structure in the solution.

Table 4: Number of connected components of μ_X at selected α-levels, for $\varepsilon = 0$ and $\varepsilon = 0.45$.

Equation	$\varepsilon = 0$			$\varepsilon = 0.45$		
	$\alpha = 0.05$	0.16	0.40	$\alpha = 0.05$	0.16	0.40
$x^3 - 3x$	2	2	2	2	3	2
$\sin(x)$	4	8	8	4	8	8
$x^2 + x$	2	2	2	2	4	2
$x^2 + 3x$	2	2	2	2	4	2
e^x	1	1	1	1	2	1
$x + x^3$	1	1	1	1	2	1

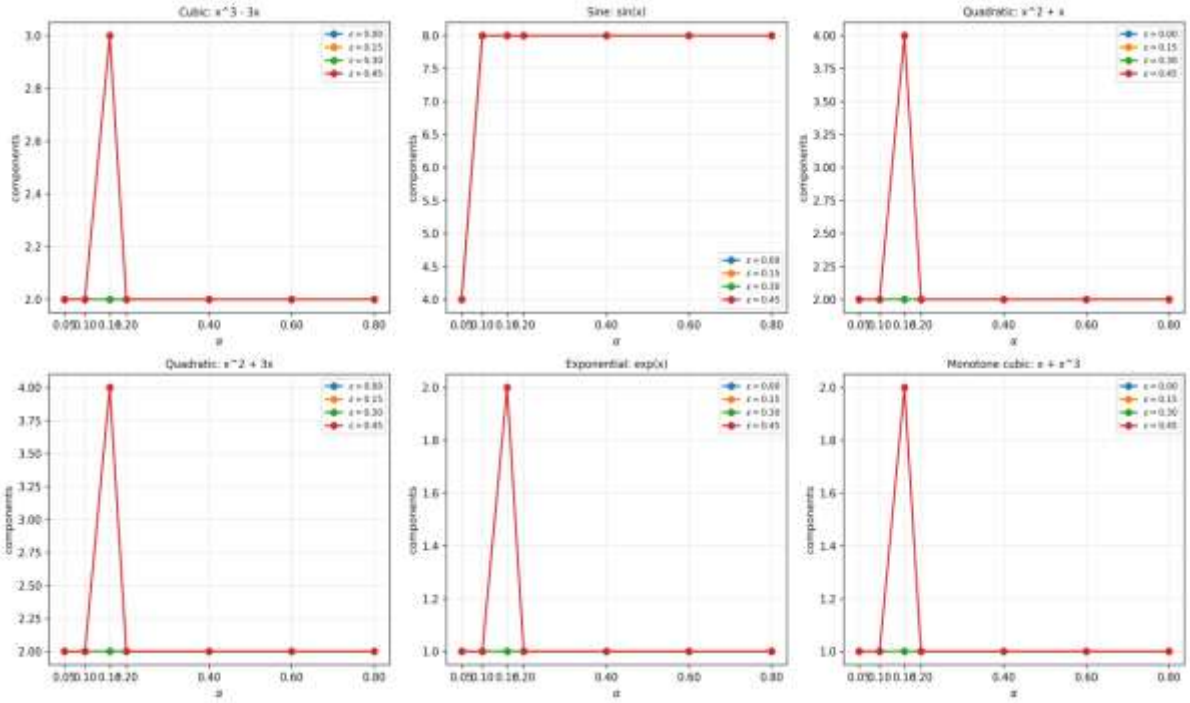


Figure 4: Number of connected components of the pullback membership $\mu_X(x)$ as a function of α, for all six equations and all four values of ε. At $\alpha = 0.16$ and $\varepsilon = 0.45$, four of six equations exhibit additional components.

Automated validation summary

The complete validation suite consists of 168 alpha-cut classifications (all reliable) and 24 COG comparisons (Simpson vs. adaptive quadrature, all with differences below 10^{-10}). All classifications passed the grid-refinement and boundary checks.

Discussion and Conclusion

The results of Section 4 establish a three-level structural classification of fuzzy algebraic equations:

1. Strong class: equations with asymmetric branches and topological changes. COG amplification is significant (-15.81% for the cubic equation).
2. Degenerate class: equations with symmetric branches (invariant by Theorem 3.1) or monotone functions. COG amplification is zero or negligible (0.00% and -0.05%).
3. Weak class: equations with a single smooth branch. COG amplification is negligible (-0.12% and $+0.20\%$).

The invariance theorem (Theorem 3.1) provides a rigorous explanation for the degenerate class: any equation of the form $x^2 + bx = \tilde{A}$ has COG $= -b/2$, regardless of the shape of $\tilde{A}$. This suggests that non-convexity amplification is not a universal phenomenon but a structural property of the algebraic equation. The alpha-cut analysis (Table 4) provides additional insight: at $\varepsilon = 0.45$, the number of connected components of the solution increases at $\alpha = 0.16$ for four of the six equations. This is a direct consequence of the AFRN's critical threshold $\varepsilon_c \approx 0.22$: above this threshold, the AFRN itself has disconnected α-cuts, and this disconnection propagates through the algebraic equation. The interaction between the non-convexity of $\tilde{A}$ and the algebraic structure of f determines the resulting amplification.

Future work

Several directions are worth pursuing. First, the structural classification should be extended to polynomial equations of higher degree and to transcendental equations. Second, the invariance theorem should be generalized to equations of the form $g(x) = h(\tilde{A})$ for suitable functions g and h. Third, the pullback method should be compared with the extension principle approach of [1] on a range of benchmark problems. Finally, applications to parameter identification and inverse problems in engineering should be explored.

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