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Analytic solutions of the classical Riccati equation and their structural isomorphism to the continuous time algebraic matrix Riccati equation

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الحلول التحليلية لمعادلة ريكاتي الكلاسيكية وتماثلها البنيوي مع معادلة ريكاتي المصفوفية ذات الزمن المستمر

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Abstract:
This paper presents a comparative analytical study of traditional solution methods for the first order nonlinear Riccati differential equation. Given the significant challenges posed by nonlinear differential equations particularly due to their quadratic nonlinearities this study focuses on one of the most prominent examples: the Riccati equation. Over the past few decades, various analytical methods have been developed to solve this equation, each method possesses its own distinct methodology, advantages, computational complexity, and scope of application. These methods include transforming the Riccati equation into a second order linear differential equation and reducing it to the Bernoulli form, a technique that relies on the availability of a known particular solution. The Riccati equation plays a prominent role in modern control theory, serving as the mathematical foundation for optimal control. A comparative analysis of the two methods covering both mathematical and practical aspects will be conducted to establish the connections between them. The explicit mathematical link between standard solutions and the matrix Riccati equation used in Linear Quadratic Regulator (LQR) systems will be identified. The study will integrate pure traditional mathematics with modern engineering applications in control theory.

Keywords: Riccati differential equation, Bernoulli reduction, second order linear, Algebraic matrix Riccati equation (AMRE), optimal control theory, structural isomorphism

الملخص
تقدم هذه الورقة دراسة تحليلية مقارنة لطرق الحل التقليدية للمعادلة التفاضلية غير الخطية من الرتبة الأولى، المعروفة بمعادلة ريكاتي. ونظراً للتحديات الكبيرة التي تفرضها المعادلات التفاضلية غير الخطية لا سيما تلك الناجمة عن وجود حدود تربيعية غير خطية- تركز هذه الدراسة على أحد أبرز أمثلتها، وهي معادلة ريكاتي. فعلى مدى العقود القليلة الماضية، طُوّرت طرق تحليلية متنوعة لحل هذه المعادلة، حيث تتميز كل منها بمنهجيتها الخاصة، ومزاياها، ودرجة تعقيدها الحسابي، ونطاق تطبيقها. وتشمل هذه الطرق تحويل معادلة "ريكاتي" إلى معادلة تفاضلية خطية من الرتبة الثانية، أو اختزالها إلى صيغة بيرنولي؛ وهي تقنية تعتمد على توفر حل خاص معلوم للمعادلة. وتؤدي معادلة ريكاتي دوراً محورياً في نظرية التحكم الحديثة، إذ تُعد الأساس الرياضي للتحكم الأمثل. وستتناول هذه الدراسة تحليلاً مقارناً لهاتين الطريقتين، من الجانبين الرياضي والعملي، لتوضيح الروابط القائمة بينهما. كما سيتم تحديد الرابط الرياضي الصريح بين الحلول القياسية لهذه المعادلة وبين

معادلة ريكاتي المصفوفية المستخدمة في أنظمة المُنظِّم الخطي التربيعي. وبذلك، تجمع الدراسة بين الرياضيات التقليدية البحتة والتطبيقات الهندسية الحديثة في مجال نظرية التحكم.

الكلمات المفتاحية: معادلة ريكاتي التفاضلية، اختزال برنولي، معادلة خطية من الرتبة الثانية، معادلة ريكاتي المصفوفية الجبرية، نظرية التحكم الأمثل، التشاكل البنيوي

Introduction

The Riccati equation is one of the most famous and widely studied nonlinear ordinary differential equations in mathematical analysis. It emerged in the 18th century, introduced by the Italian scholar Jacopo Riccati, and was named after him[1,2]. Its significance extends beyond pure mathematics; it plays a prominent role in engineering applications particularly in modern control theory appearing in problems related to optimal control, the linear-quadratic regulator (LQR), the Kalman filter, stochastic control, and dynamic games [3,4] . The standard Riccati equation can be expressed in the following form:

$$\frac{dy}{dx} = p(x) + Q(x)y + R(x)y^2 \qquad (1)$$

where $p(x), Q(x), R(x)$ are continuous real valued functions; if $R(x) = 0$,the equation becomes a first order linear ordinary differential equation, and when $p(x) = 0$ it takes the form of the classical Bernoulli equation [5,6]. Despite advancements in numerical methods and algorithms for solving the Riccati equation, analytical approaches remain a subject of active research and hold significant importance in the scientific literature; they reveal the equation's mathematical structure and the qualitative properties of nonlinear systems insights that numerical solutions do not provide [7] .The most prominent of these methods include transforming it into a second-order linear differential equation, as well as using a particular solution to convert it into a Bernoulli equation; each method involves specific mathematical steps and conditions for application, making them suitable approaches for finding a solution [6,8]. With the diversity of solution methods, comparative studies have become increasingly important and essential; they go beyond merely presenting various approaches to offer an understanding of the underlying mathematical and to clarify the advantages and limitations that dictate the choice of one method over another. these studies bridge the gap between traditional analytical techniques and modern applications in the field of control[3,7].

This paper presents a comparative, analytical review of two reduction methods for solving the Riccati equation and demonstrates a structural analogy between them and the matrix algebraic Riccati equation used in optimal control system design, focusing on their mathematical foundations and applications in control theory[4,9]. The paper serves as a concise scientific reference for researchers and those interested in the Riccati equation and its solution methods.

1 -Classical Analytical Solution Methods:

To understand the mathematical structure of the first order nonlinear Riccati equation, one must analyze specific variable transformations that convert its quadratic nonlinearity into a linear form. This section presents two analytical methods used to solve the Riccati equation:

1.1 Reduction to a second order linear differential equation:

The standard Riccati equation is related to the theory of second order linear differential equations when $R(x) = 0$. By applying the appropriate substitution[5,6]

$$y(x) = -\frac{1}{R(x)u(x)}\frac{du}{dx} \qquad (2)$$

where $u(x)$ is a twice differentiable dependent variable.

to eliminate the quadratic nonlinearity in equation (1), the problem is transformed into a higher-order functional space [1,8].We derive equation (2) with respect to x, we obtain the equation

$$\frac{dy}{dx} = -\frac{u''}{R(x)u} + \frac{R'(x)u'}{R(x)^2u} + \frac{(u')^2}{R(x)u^2} \qquad (3)$$

We substitute equations (2) and (3) into equation (1) and expand the quadratic term, causing the non linear terms to vanish [8], we then rearrange the equation and perform the necessary steps to transform it into a second order linear ordinary differential equation[5,8]:

$$u'' - \left(Q(x) + \frac{R'}{R(x)}\right)u' + p(x)R(x)u = 0 \quad (4)$$

This equation is solved based on the coefficients $Q(x), R(x)$ and $p(x)$,whether they are constants or simple functions using methods such as the integrating factor or a change of variables; if the coefficients are variable, the equation transforms into forms involving special functions (such as Airy or Bessel functions), which have their own specific solution methods[7].

1.2 Particular solution transformation:

If we have a known particular solution $y_1(x)$, we can obtain the general solution to the Riccati equation by isolating the perturbation state $u(x)$, [6,8], taking the following form:

$$y(x) = y_1(x) + u(x) \quad (5)$$

and by deriving equation (5) $y'(x) = y_1'(x) + u'(x)$ (6) ,

deriving equation (5), and since $y_1(x)$ is a solution that satisfies equation (1), then

$$y_1'(x) = p(x) + Q(x)y_1(x) + R(x)y_1^2(x) \quad (7)$$

Substituting equations (6) and (7) into equation (1), it becomes a classical Bernoulli differential equation in terms of $u(x)$, [5,8]:

$$u'(x) - [Q(x) + 2R(x)y_1(x)]u(x) = R(x)u^2(x) \quad (8)$$

by dividing Equation (8) by $u^2(x)$ and setting $v = \frac{1}{u(x)}$, the equation transforms into a first order linear equation of the following form:

$$v'(x) + [Q(x) + 2R(x)y_1(x)]v(x) = -R(x) \quad (9)$$

Equation 9 is then solved using an integrating factor [6].

1.3 Illustrative example:

Solve the following Riccati equation: $\frac{dy}{dx} = 1 - 2y - y^2$ (10)

where $p(x) = 1, Q(x) = -2$, $R(x) = 1$,

1.3.1 Solution via second order linear reduction:

using conversion equation (2), we obtain $y(x) = \frac{u'}{u}$ (11)

by substituting the coefficients into Equation 4, we obtain a second order linear equation:
$u''(x) + 2u'(x) + u(x) = 0$ (12)

by solving it, we obtain the general solution: $y(x) = 1 - \frac{1}{x-c}$ (13)

1.3.2 The solution involves transforming it into a Bernoulli equation using the particular solution

We have a well known, specialized solution $y_1(x) = 1$

we take a known particular solution $y_1(x) = 1$ and substitute it into transformation equation (5); differentiating with respect to x yields the equation

$$u'(x) = u^2(x) \qquad (14),$$

which is a Bernoulli equation. Solving this using the transformation $v(x) = \frac{1}{u(x)}$ results in a general solution that is mathematically consistent with equation (13).

Physical interpretation of the general solution in the steady state*:*

In the equation (13), the disturbance term $u(x) = \frac{1}{x-c}$ eventually decays over time $x \to \infty$, allowing the system to reach the desired state at $y_{desired} = 1$. This state coincides with the actual system state $y(x)$, (i.e.,$y_{desired} = y(x) = 1$), the system experiences no deviation and is completely stable.

2. Qualitative comparative analysis:

In both methods, the goal is to reduce nonlinearity to linearity. We will encounter various mathematical challenges; the following table summarizes a comparison between two analytical methods for solving the standard Riccati differential equation, based on specific evaluation criteria:

Basis of comparison (evaluation criterion)	**Reduction to a second order linear equation, Section (2.1).**	**Transformation into a Bernoulli equation using a known particular solution, Section (2.2).**
Workflow	Transforming a first order nonlinear equation into a second-order linear differential equation[5,8].	The original variable $y(x)$ is expressed as the sum of two terms a particular solution and a perturbation term which is then isolated and converted into a first order linear equation that is easy to solve[6,8].
Advance information required	None; it only requires knowledge of the function coefficients in the equation[5].	It requires knowledge of the particular solution $y_1(x)$, [5].
Type of resulting equation.	A second order linear homogeneous differential equation [8].	A Bernoulli differential equation obtained via transformation[5,6]
Algebraic complexity.	It may yield variable coefficients that require knowledge of special functions[8].	It is solved using the Bernoulli transformation and an integrating factor[5,6].
Scope of application.	It is broad and applicable to all continuous Riccati equations[5,8].	It is restricted to systems with distinguishable roots that is, systems where a particular solution is easily guessed or identified[6,8].
Physical meaning in control.	It represents the total transformation of the system's state and its overall dynamics[5,9].	It involves isolating the perturbation term, studying it, and determining the extent of its decay over time[4,8].

3. Applications in Control Theory: Structural Symmetry of the Matrix Riccati Equation (AMRE):

In modern control systems such as those found in aircraft and robots multiple variables are involved; consequently, matrices are used to represent the system's state. To ensure stability, the scalar Riccati equation defined in a one dimensional space is extended to the multidimensional Algebraic Riccati Equation[3,4], which governs control system stability and optimal control design[4,9,10].

To achieve optimal feedback gains, control designers employed a symmetric positive definite matrix P solution satisfying the continuous time algebraic Riccati equation[4,9],

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad (15)$$

upon comparing Equation (15) with Equation (1) , we observe a structural similarity[3,4,9]:

- The quadratic nonlinear term in the control equation $PBR^{-1}B^T P$ is equivalent to the term $R(x)y^2$ in the standard Riccati equation, and the coefficient (set of control matrices) $BR^{-1}B^T$ corresponds to the coefficient $R(x)$.
- The linear term of matrix $A^T P + PA$ corresponds to the term $Q(x)y$ in the standard equation.
- The independent (constant) term: the weight matrix $Q(x)$ corresponds exactly to the standard function $P(x)$.

The algebraic properties of the Riccati equation in one dimension are the same as those that determine the existence and uniqueness of solutions and the stability of complex dynamical systems.

Conclusion: This paper presented an analytical study comparing two fundamental methods for solving the standard Riccati differential equation: the first involves reduction to a second-order linear equation, while the second entails transformation into a Bernoulli equation using a known particular solution. The comparison revealed that the first method is distinguished by its broad applicability and wide scope, whereas the second, while algebraically simple to solve, is limited in scope.

The two approaches have been linked via optimal control theory through a structural analogy between the scalar Riccati equation and the algebraic matrix Riccati equation. This connection demonstrated that the existence of a positive definite solution matrix P which ensures control system stability is structurally equivalent to the existence of specific solution conditions in the scalar Riccati equation; understanding the algebraic properties of the scalar version of the equation guarantees the stability of complex dynamic systems prior to the implementation of algorithms and numerical approximations.

References:

[1] Reid, W. T. (1972). *Riccati differential equations*. Academic Press.

[2] Ince, E. L. (1956). *Ordinary differential equations*. Dover Publications.

[3] Ogata, K. (2010). *Modern control engineering* (5th ed.). Prentice Hall.

[4] Kirk, D. E. (2004). *Optimal control theory: An introduction*. Dover Publications.

[5] Coddington, E. A., & Levinson, N. (1955). *Theory of ordinary differential equations*. McGraw-Hill.

[6] Boyce, W. E., DiPrima, R. C., & Meade, D. B. (2017). *Elementary differential equations and boundary value problems* (11th ed.). John Wiley & Sons.

[7] Evans, L. C. (2010). *Partial differential equations* (2nd ed.). American Mathematical Society.

[8] Zill, D. G. (2018). *A first course in differential equations with modeling applications* (11th ed.). Cengage Learning.

[9] Lewis, F. L., Vrabie, D. L., & Syrmos, V. L. (2012). *Optimal control* (3rd ed.). John Wiley & Sons.

[10] Anderson, B. D. O., & Moore, J. B. (2007). *Optimal control: Linear quadratic methods*. Dover Publications.

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