



Integral Averages of Generalization of the Operator -Valued Poisson Kernel and Poisson Kernel Angle Type Formula

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متوسط التكامل لتعميم نواة بواسون بقيم مؤثرية والصيغة الزاوية لنواة بواسون

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Abstract:

This study investigates the generalization of the Poisson Kernel and the operator-valued Poisson Kernel from a single variable to two variables, including their integral formulas. The Leibniz rule is utilized to present the average for the extension of the operator-valued Poisson kernel. Furthermore, the angle type formula for the Poisson kernel and its corresponding integral formula, obtained by replacing the single variable with two variables in the original definition, are derived.

Keywords: Poisson Kernel, Generalization of Poisson Kernel, Operator-valued Poisson Kernel.

المخلص

تبحث هذه الدراسة في تعميم نواة بواسون وتعميم نواة بواسون بقيم مؤثرية من بارامتر واحد إلى بارامترين، بما في ذلك الصيغ التكاملية الخاصة بهما. وتستخدم قاعدة لبنز لإيجاد قيمة متوسط التكامل لتعميم نواة بواسون بقيم مؤثرية. وعلاوة على ذلك، يتم اشتقاق الصيغة من النوع الزاوي لنواة بواسون وصيغتها التكاملية المناظرة، وذلك من خلال استبدال متغير واحد بمتغيرين في التعريف الأصلي.

الكلمات المفتاحية: نواة بواسون، تعميم نواة بواسون، نواة بواسون بقيم مؤثرية.

Introduction

In potential theory, the Poisson kernel is an integral kernel that plays a fundamental role in solving the two-dimensional Laplace equation subject to Dirichlet boundary conditions on the unit disk. In complex analysis, the Poisson kernel is particularly important because integrating it against a function on the unit circle yields a harmonic extension of that function to the unit disk. Since harmonic functions are related to meromorphic functions, the two-dimensional Dirichlet problem is essentially equivalent to finding a meromorphic extension of a function defined on the boundary. Moreover, in this study, Galbraith and Green (1947) obtained the mean value of the Poisson kernel $P(\theta, r)$, where r is a real parameter satisfying $|r| < 1$. Subsequently, Haruki and Rassias (1997) introduced a two-parameter generalization of the Poisson kernel, denoted by $Q(\theta, a, b)$, where a and b are complex parameters satisfying $|a| < 1$ and $|b| < 1$, and established an integral formula for this generalized kernel. Later, Kim (1999) provided a solution to an open problem posed by Haruki and Rassias concerning the powers of $Q(\theta, a, b)$.

Furthermore, (Chalendar, 2003) introduced the operator- valued Poisson Kernel denoted $K_{r,t}(T)$. Then, (Bulut, 2011) proposed a generalization of the operator- valued Poisson Kernel $Q_{a,b,t}(T)$ and studied its corresponding integral representation. In addition, (Al-Sharif et al., 2013) discussed the value of the integral $\int_0^{2\pi} Q_{a,b,t}(T)^{n+1}$, for all $n = 0, 1, 2, 3, \dots$, under the special case when $b = \bar{a}$ and the operator $(I - ae^{it}T^*)$ is self-adjoint. These results raise the question of whether analogous integral average formulas can be established without imposing this restrictive assumption. Moreover, previous studies have generalized the Poisson kernel with respect to the parameter r leading to generalizations involving two, four, and six variables (Bulut, 2006; Haruki & Rassias, 1997) What if the generalization of Poisson's kernel is for angle θ , and what about the integral in this new generalization?

In this paper, we investigate integral averages of a generalization of the operator-valued Poisson Kernel. Furthermore, we introduce a new generalization of the Poisson Kernel by replacing θ in $P(\theta, r)$ and $Q(\theta, a, b)$ with θ_1, θ_2 and evaluate their corresponding integral formulas, examining the properties of these generalized kernels.

The paper is organized as follows: Section 2 presents the Poisson kernel, its generalization, and related theorems. Section 3 includes definitions and theorems related to the operator-valued Poisson kernel. Section 4 presents the main results of the generalization of the operator-valued Poisson kernel and the Poisson Kernel.

2. Preliminaries:

The Poisson Kernel in two dimensions $p(\theta, r)$ is defined by

$$p(\theta, r) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2} = \frac{1 - r^2}{(1 - re^{i\theta})(1 - re^{-i\theta})}. \quad (1)$$

Here r is a real parameter satisfying $|r| < 1$, and $-\infty < \theta < \infty$.

Theorem 2.1 (Galbraith & Green, 1947)

$$\frac{1}{2\pi} \int_0^{2\pi} P(\theta, r) d\theta = 1. \quad (2)$$

Where r is a real parameter satisfying $|r| < 1$.

The Generalization of the Poisson Kernel $Q(\theta, a, b)$ is given in the following definition

Definition 2.1 (Haruki & Rassias, 1997)

$$Q(\theta, a, b) = \frac{1 - ab}{(1 - ae^{i\theta})(1 - be^{-i\theta})}, \quad (3)$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Theorem 2.2 (Haruki & Rassias, 1997)

$$\frac{1}{2\pi} \int_0^{2\pi} Q(\theta, a, b) d\theta = 1 \quad (4)$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Remark 1: setting $a = r$ and $b = r$ in (3) and (4) yields (3) and (4) are generalizations of (1) and (2) respectively.

Theorem 2.3 (Kim, 1999)

$$\frac{1}{2\pi} \int_0^{2\pi} Q(\theta, a, b)^{n+1} d\theta = \sum_{k=0}^n \frac{(2n-k)!}{(k)!((n-k)!)^2} \left(\frac{ab}{1-ab} \right)^{n-k}, \quad n = 0, 1, 2, 3, \dots \quad (5)$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

3. The operator- valued Poisson Kernel

Let H be a complex Hilbert space and $L(H)$ be the algebra of all linear bounded mappings from H to H . For any operator $T \in L(H)$, its spectrum $\sigma(T)$ is the nonempty compact subset of the complex plane $\mathbb{C}$ consists of all scalars $\lambda \in \mathbb{C}$ for which the operator $T - \lambda I$ is non-invertible in $L(H)$, I represents the identity operator on H and D is the unit disk in $\mathbb{C}$.

Definition 3.1 (Chalendar, 2003) Suppose $T \in L(H)$ such that $\sigma(T) \subset \bar{D}$ and for $re^{it} \in D$. The operator- valued Poisson Kernel is defined by

$$K_{r,t}(T) = (I - re^{it}T^*)^{-1} + (I - re^{-it}T)^{-1} - I.$$

Theorem 3.1 (Chalendar, 2003) Assume $T \in L(H)$ such that $\sigma(T) \subset \bar{D}$, then

$$\frac{1}{2\pi} \int_0^{2\pi} K_{r,t}(T) dt = I,$$

where r is a real parameter satisfying $|r| < 1$.

The following Theorem presents the generalization of the operator-valued Poisson kernel.

Definition 3.2 (Bulut, 2011) Let $T \in L(H)$ such that $\sigma(T) \subset \bar{D}$ and for $re^{it} \in D$, we have

$$Q_{a,b,t}(T) = (I - ae^{it}T^*)^{-1} + (I - be^{-it}T)^{-1} - I,$$

Theorem 3.2 (Bulut, 2011) Let $T \in L(H)$ such that $\sigma(T) \subset \bar{D}$ and for $re^{it} \in D$, then,

$$\frac{1}{2\pi} \int_0^{2\pi} Q_{a,b,t}(T) dt = I. \quad (6)$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Theorem 3.3 (Al-Sharif et al., 2013) Let $T \in L(H)$ such that $\sigma(T) \subset \bar{D}$, $b = \bar{a}$ and $(I - ae^{it}T^*)$ is self-adjoint, we have

$$\frac{1}{2\pi} \int_0^{2\pi} Q_{a,b,t}(T)^{n+1} dt = \sum_{k=0}^{n+1} \sum_{l=0}^k \binom{n+1}{k} \binom{k}{l} (-I)^l,$$

such that $n = 0, 1, 2, 3, \dots$, and a, b are complex parameters satisfying $|a| < 1$.

4. Main Results of Generalization of Poisson Kernel

4.1 Integral averages of generalization of the operator- valued Poisson Kernel

In this section, the value of the integral $\int_0^{2\pi} Q_{a,b,t}(T)^{n+1} dt$, for all $n = 0, 1, 2, 3, \dots$ is defined without the special case in **Theorem 3.3**.

Theorem 4.1.1 Let $T \in L(H)$ such that $\sigma(T) \subset \bar{D}$, then

$$\frac{1}{2\pi} \int_0^{2\pi} Q_{a,b,t}(T)^{n+1} dt = \sum_{k=0}^n \frac{(n+k)!}{(n-k)!(k!)^2} \left(\frac{abTT^*}{I - abTT^*} \right)^k, \quad n = 0, 1, 2, 3, \dots, \quad (7)$$

Proof:

Suppose

$$I_n = \frac{1}{2\pi} \int_0^{2\pi} Q_{a,b,t}(T)^{n+1} dt = ((I - ae^{it}T^*)^{-1} + (I - be^{-it}T)^{-1} - I)^{n+1}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{I - abTT^*}{(I - ae^{itT^*})(I - be^{-itT})} \right)^{n+1} dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{((I - abTT^*)/(I - ae^{itT^*}))^{n+1}}{(I - be^{-itT})^{n+1}} dt
\end{aligned}$$

By the change of variable $z = e^{it}$ and setting

$$f(z) = \left(\frac{I - abTT^*}{I - azT^*} \right)^{n+1} z^n \quad (8)$$

$$I_n = \frac{1}{2\pi} \int_{|z|=1} \frac{f(z)}{(z - bT)^{n+1}} dz,$$

where the integral of the function $f(z)$ is in the positive direction along the unit circle $|z| = 1$.

Hence, by the integral formula, since $f(z)$ is an analytic function in $|z| \leq 1$, we obtain

$$I_n = \frac{f^{(n)}(bT)}{n!}$$

Now, to find I_n , we will use the Leibniz rule.

Assume

$$l(z) = z^n$$

$$m(z) = (I - azT^*)^{-(n+1)} \quad (9)$$

Then by (8) and (9), we obtain

$$f(z) = (I - abTT^*)^{n+1} m(z) l(z) \quad (10)$$

By applying the Leibniz rule to (10), we have

$$\begin{aligned}
f^n(z) &= (I - abTT^*)^{n+1} (lm)^n(z) \\
&= (I - abTT^*)^{n+1} \sum_{k=0}^n \binom{n}{k} l^{n-k}(z) m^k(z) \\
&= n! (I - abTT^*)^{n+1} \sum_{k=0}^n \frac{(n+k)!}{(n-k)!(k!)^2} (I - azT^*)^{-(n+1+k)} (azT^*)^k
\end{aligned} \quad (11)$$

If $z = bT$ in (11), then

$$I_n = \frac{f^n(bT)}{n!} = \sum_{k=0}^n \frac{(n+k)!}{(n-k)!(k!)^2} \left(\frac{abTT^*}{I - abTT^*} \right)^k.$$

Remark 2: By choosing $n = 0$ in (7), we obtain (6).

Corollary: Let $T \in L(H)$ such that $\sigma(T) \subset D^-$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} Q_{a,b,t}(T)^2 dt = \frac{I + abTT^*}{I - abTT^*},$$

Remark 3: When we take $TT^* = I$ in (7), we obtain that (5) is the scalar form of (7), in other words, **Theorem 2.3** is a special case of **Theorem 4.1.1**

4.2. Poisson Kernel Angle Type Formula

In this section, we present the definition of the Poisson integral angle type formula and get some new results on the integral formula of $P(\theta_1, \theta_2, r)$ and $Q(\theta_1, \theta_2, a, b)$.

Analogous to the Poisson Kernel $P(\theta, r)$ in (1), we define the Poisson kernel of angle type as follows:

Definition 4.2.1

$$\begin{aligned} P(\theta_1, \theta_2, r) &= \frac{1}{(1 - re^{i\theta_1})} + \frac{1}{(1 - re^{-i\theta_2})} - 1 \\ &= \frac{1 - r^2 e^{i(\theta_1 - \theta_2)}}{(1 - re^{i\theta_1})(1 - re^{-i\theta_2})}, \end{aligned} \quad (12)$$

where r is a real parameter satisfying $|r| < 1$, $-\infty < \theta_1, \theta_2 < \infty$.

The integral formula for of $P(\theta_1, \theta_2, r)$ is given in the following theorem.

Theorem 4.2.1

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} P(\theta_1, \theta_2, r) d\theta_1 d\theta_2 = 1, \quad (13)$$

Proof: sitting $z = e^{i\theta_1}$ and $w = e^{i\theta_2}$ in the following integral, we get

$$\begin{aligned} &\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} P(\theta_1, \theta_2, r) d\theta_1 d\theta_2 \\ &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2 e^{i(\theta_1 - \theta_2)}}{(1 - re^{i\theta_1})(1 - re^{-i\theta_2})} d\theta_1 d\theta_2 \\ &= \frac{1}{4\pi^2} \int_{|z|=1} \int_{|w|=1} \frac{1 - r^2 zw}{(1 - rz)(1 - rw)zw} dz dw \\ &= \frac{1}{4\pi^2} \int_{|z|=1} \frac{1}{(1 - rz)z} dz \int_{|w|=1} \frac{1}{(1 - rw)w} dw \\ &\quad - \frac{r^2}{4\pi^2} \int_{|z|=1} \frac{1}{(1 - rz)} dz \int_{|w|=1} \frac{1}{(1 - rw)} dw \end{aligned}$$

By the Residue Theorem, we have

$$\begin{aligned} \int_{|z|=1} \frac{1}{(1 - rz)} dz &= \int_{|w|=1} \frac{1}{(1 - rw)} dw = 0 \\ \int_{|z|=1} \frac{1}{(1 - rz)z} dz &= \int_{|w|=1} \frac{1}{(1 - rw)w} dw = 1 \end{aligned}$$

Then

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} P(\theta_1, \theta_2, r) d\theta_1 d\theta_2 = 1.$$

Remark 4: sitting $\theta_1 = \theta_2 = \theta$ in (12), yields (1).

$$P(\theta_1, \theta_2, r) = p(\theta, r).$$

Analogous to the generalization of the Poisson Kernel $Q(\theta, a, b)$ in (3), we define a generalization of the Poisson kernel of angle type as follows:

Definition 4.2.2

$$Q(\theta_1, \theta_2, a, b) = \frac{1}{(1 - ae^{i\theta_1})} + \frac{1}{(1 - be^{-i\theta_2})} - 1 = \frac{1 - abe^{i(\theta_1 - \theta_2)}}{(1 - ae^{i\theta_1})(1 - be^{-i\theta_2})} \quad (14),$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

The integral formula for $Q(\theta_1, \theta_2, a, b)$ is given in the following theorem.

Theorem 4.2.2

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} Q(\theta_1, \theta_2, a, b) d\theta_1 d\theta_2 = 1 \quad (15)$$

Proof:

The proof is analogous to the proof of Theorem 4.1.

Remark 5:

- i. setting $\theta_1 = \theta_2 = \theta$ in (14), yields (3)

$$Q(\theta_1, \theta_2, a, b) = Q(\theta, a, b).$$
- ii. taking $a = b = r$ in (14) and (15), we obtain (14) and (15) are generalizations of (12) and (13), respectively.

Conclusion

This work investigates the Poisson Kernel and operator-valued Poisson Kernel. It establishes comprehensive results concerning the integral averages of a generalized operator-valued Poisson Kernel involving the power of $Q_{a,b,t}(T)$. Furthermore, a new angle type formula for the Poisson kernel is provided, and two new angle type formulas are introduced. The corresponding integral formulas are also derived and evaluated. These definitions and theorems contribute to the further development of the theorems of Poisson Kernels and highlight their practical applications.

Compliance with ethical standards*Disclosure of conflict of interest*

The authors declare that they have no conflict of interest.

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