

Traveling Wave Solutions of Murray's Blood-Flow Equation and Its Modified Forms via the Variation of (G'/G) -Expansion Method: Analytical Derivation and a Clinical Application to Coronary Artery Diameter Assessment

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حلول الموجات المنتقلة لمعادلة موراي لتدفق الدم وصيغها المعدلة باستخدام طريقة مفكوك (G'/G) المطورة: اشتقاق تحليلي وتطبيق سريري لتقييم قطر الشريان التاجي

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Abstract:

Murray's law, the classical minimum work principle governing the optimal caliber of branching blood vessels, provides a physiologically grounded target for the baseline (steady, unforced) state of one-dimensional reaction diffusion models of blood flow. In this work, the Variation of the (G'/G) -Expansion Method an analytical scheme that expresses the traveling-wave amplitude as a joint polynomial in two auxiliary Riccati-type functions, G'/G and F'/F is applied to Murray's nonlinear partial differential equation and to two of its structural modifications: a cubic-reaction variant and a variant extended by a quadratic dissipation (gradient) term. For all three formulations, the balancing procedure and the resulting over determined algebraic systems, solved with a computer algebra system, converge on the same non-trivial equilibrium traveling-wave solution, $u = \delta_2/\delta_3$, independent of the specific nonlinear closure used. This equilibrium ratio is then linked to Murray's power law, $Q \propto D^p$, using the clinically derived exponent $p = 2.15$ reported for human epicardial coronary arteries, yielding a reference value of 14.78 that reproduces the normal left anterior descending (LAD) artery diameter of 3.5 mm. A diagnostic look-up table connecting this reference value to LAD narrowing and dilation is presented. The invariance of the equilibrium solution across three structurally distinct forms of Murray's equation is interpreted as evidence of the robustness of both the underlying physiological model and the solution methodology, and the paper concludes by outlining how the same reference value framework could support noninvasive screening of coronary caliber.

Keywords: Traveling Wave Solutions, (G'/G) -Expansion Method, Murray's Equation, Murray's Law, Coronary Artery Diameter, Nonlinear Partial Differential Equations And Mathematical Physiology

المخلص

يُقدم قانون "موراي" (Murray's law) -وهو المبدأ الكلاسيكي للحد الأدنى من الشغل الذي يحكم القطر الأمثل للأوعية الدموية المتفرعة مرجعاً يستند إلى أسس فسيولوجية لتحديد الحالة الأساسية (المستقرة وغير القسرية) في نماذج "التفاعل والانتشار" أحادية البعد لتدفق الدم. وفي هذا العمل، تُطبّق "طريقة مفكوك (G'/G)"- المطورة وهي أسلوب تحليلي يعبر عن سعة الموجة المنتشرة كمتعددة حدود مشتركة بدلالة دالتين مساعدتين من نوع "ريكاتي" (Riccati)، وهما F'/F و G'/G على معادلة "موراي" التفاضلية الجزئية غير الخطية، وكذلك على صيغتين معدلتين بنيوياً لها: صيغة تتضمن تفاعلاً من الدرجة الثالثة (تكعيبياً)، وصيغة موسعة بإضافة حد تبديد (تدرج) من الدرجة الثانية (تربيعي). وبالنسبة للصيغ الثلاث، أدت إجراءات الموازنة والأنظمة الجبرية الزائدة عن الحاجة (over-determined systems) الناتجة عنها والتي حُلّت باستخدام نظام للجبر الحاسوبي إلى الوصول للحل نفسه المتمثل في موجة منتشرة متوازنة وغير تافهة (non-trivial)، وهي $u = \delta_2/\delta_3$ ، وذلك بمعزل عن نموذج الإغلاق غير الخطي المحدد المستخدم. ثم رُبطت نسبة التوازن هذه بقانون القوى الخاص بـ "موراي" ($Q \propto D^p$) باستخدام الأس $p = 2.15$ المستمد سريرياً والمُسجّل للشرابين الناجية البشرية الموجودة على سطح القلب (epicardial coronary arteries)؛ مما أسفر عن قيمة مرجعية قدرها 14.78 تعيد إنتاج القطر الطبيعي للشريان الأمامي النازل الأيسر (LAD) البالغ 3.5 مم. كما يقدم البحث جدولاً مرجعياً تشخيصياً يربط هذه القيمة المرجعية بحالات تضيق وتوسع الشريان الأمامي النازل الأيسر (LAD). ويُفسّر ثبات حل التوازن عبر الأشكال الثلاثة المختلفة بنيوياً لمعادلة "موراي" على أنه دليل على متانة كل من النموذج الفسيولوجي الأساسي ومنهجية الحل، ويختتم البحث بتوضيح كيف يمكن لإطار العمل القائم على هذه القيمة المرجعية أن يدعم عمليات الفحص غير الجراحي لقطر الشرايين الناجية.

الكلمات المفتاحية: حلول الموجة المنتقلة، طريقة مفكوك (G'/G) المطورة، معادلة موراي، قانون موراي، قطر الشريان الناجي، المعادلات التفاضلية الجزئية غير الخطية، والفسيولوجيا الرياضية.

1. Introduction

Nonlinear partial differential equations (NPDEs) are the natural language of biological transport phenomena, where diffusion, advection and nonlinear reaction terms interact to produce traveling fronts, pulses and equilibria with direct physiological meaning. Among the many analytical techniques developed to construct closed-form traveling-wave solutions of such equations the tanh method, the sine-cosine method, the Exp-function method and the auxiliary-equation method the (G'/G)-expansion method, introduced by Wang, Li and Zhang (2008), has become one of the most widely used because of its algorithmic simplicity and its ability to recover hyperbolic, trigonometric and rational solution families from a single second-order auxiliary equation. Since its introduction, the method has been extended in numerous directions, including generalized, modified, extended and alternative (G'/G)-expansion schemes designed to enlarge the space of admissible solutions and to handle higher-order balancing.

The present study uses a two-function variant of this family, referred to here as the Variation of the (G'/G)-Expansion Method, in which the wave amplitude is expanded jointly in the Riccati-type quantities G'/G and F'/F , with G and F satisfying a coupled pair of first-order auxiliary equations whose closed-form solutions are the hyperbolic pairs (sech, tanh) and (csch, coth). This structure increases the number of free coefficients available at a given balancing order m , which is useful for equations such as the ones considered below whose nonlinear reaction term is not a simple monomial.

The equation chosen to illustrate the method carries a specific physiological motivation. Murray's law, first proposed by Murray (1926) on the principle of minimum work, states that the volumetric blood flow Q through a vessel scales with a power of its diameter D , $Q \propto D^p$, with $p = 3$ in Murray's original derivation. Decades of subsequent morphometric and, more recently, computational-fluid-dynamics studies have shown that the empirical exponent for the human coronary tree deviates from 3; a systematic review and meta-analysis of the coronary literature reports a pooled flow-diameter exponent of 2.39, while a physiology-based computational study using invasively validated flow measurements in human epicardial arteries identified an optimal exponent of $p = 2.15$ for absolute coronary flow. Because the left anterior descending (LAD) artery is the vessel most frequently implicated in ischemic heart disease, a mathematically explicit link between a reaction diffusion model of vessel wall dynamics and Murray's scaling law is of practical as well as theoretical interest.

This paper analyzes a one dimensional nonlinear reaction diffusion equation associated with Murray's law referred to here as Murray's equation together with two structural modifications: a modified form in which the quadratic reaction term is replaced by a cubic term, and an extended form incorporating an additional quadratic gradient (dissipation) term. Applying the Variation of the (G'/G)-expansion method to all three forms, we show that the resulting non-trivial traveling-wave solution is, in every case, the constant equilibrium $u = \delta_2/\delta_3$, and we use this invariance to argue for the robustness of the underlying physiological equilibrium. We then translate this equilibrium ratio into a clinically interpretable diameter reference value via Murray's power law with $p = 2.15$, and tabulate the resulting diagnostic ranges for LAD narrowing, normal caliber, and dilation.

The remainder of the paper is organized as follows. Section 2 describes the Variation of the (G'/G) -expansion method in general form. Section 3 applies the method to the original, modified and extended forms of Murray's equation. Section 4 develops the clinical translation of the resulting equilibrium solution via Murray's law. Section 5 discusses the results, and Section 6 concludes.

Material and methods

2.The Variation of the (G'/G) -Expansion Method

Consider a nonlinear partial differential equation in the dependent variable u and independent variables $x_1, x_2, \dots, x_n, t, P(u, u_t, u_{x_1}, u_{x_2}, u_{tt}, u_{x_1t}, u_{x_2t}, u_{x_1x_1}, u_{x_2x_2}, \dots) = 0$. (1)

Step 1 (wave transformation). The equation is reduced to an ordinary differential equation (ODE) by the traveling-wave transformation

$$u = u(x_1, x_2, \dots, x_n, t) = U(\xi) \quad \xi = \sum_{i=1}^n r_i x_i + \omega t \quad (2)$$

where $r_i (i = 1, 2, \dots, n)$ and ω are constants representing wave-number and angular-frequency-type parameters. Substitution reduces the PDE to an ODE

$$Q(U, \omega U', r_1 U', r_2 U', \omega^2 U'', r_1 \omega U'', r_2 \omega U'', r_1^2 U'', r_2^2 U'', \dots) = 0. \quad (3)$$

Step 2 (integration). If possible Eq.(3) is integrated term by term one or more times, introducing constants of integration.

Step 3 (ansatz). The solution $U(\xi)$ is expressed as a joint polynomial in $\frac{G'}{G}$ and $\frac{F'}{F}$,

$$U(\xi) = \sum_{i=0}^m a_i (G'/G)^i + \sum_{i=1}^m b_i (G'/G)^{i-1} (F'/F), \quad (4)$$

where $a_i (i = 0, 1, \dots, m), b_i (i = 1, 2, \dots, m)$ are constants to be determined and m is a positive integer fixed by the homogeneous-balance procedure. The auxiliary functions G and F satisfy the coupled first-order (Riccati-type) system

$$G'(\xi) = -G(\xi) \cdot F(\xi), \quad (5)$$

$$F'(\xi) = 1 - F(\xi)^2, \quad (6)$$

whose known closed-form solutions are the hyperbolic pairs

$$G(\xi) = \pm \operatorname{sech}(\xi), \quad F(\xi) = \tanh(\xi) \quad (7)$$

$$G(\xi) = \pm \operatorname{csch}(\xi), \quad F(\xi) = \coth(\xi), \quad (8)$$

Step 4 (balancing). The order m is fixed by balancing the highest order derivative term against the highest order nonlinear term in the transformed ODE, using the homogeneous balance rule

$$\text{If } D[U(\xi)] = m, \text{ then } D \left[U^r \left(\frac{d^q U}{d\xi^q} \right)^s \right] = mr + s(q + m) m \quad (9)$$

Step 5 (algebraic reduction). The ansatz (4) is substituted into the ODE collecting the coefficients of each power of $F = F(\xi)$ and setting every coefficient to zero yields an over determined system of nonlinear algebraic equations in $a_0, a_1, \dots, b_1, b_2, \dots$ and the wave speed parameter c (or r, v).

Step 6 (solution). The algebraic system is solved with a computer algebra system (Maple); the resulting constants are substituted back into ansatz (4) together with the closed form Riccati solutions (7),(8) to obtain the explicit traveling wave solution(s) of the original PDE.

3. Application to Murray's Equation and Its Modified Forms

3.1 The Original Murray Equation

Murray's equation, in the reaction–diffusion form used here, models the balance between diffusive, advective and nonlinear reactive contributions to one-dimensional blood-flow dynamics:

$$u_t - u_{xx} - \delta_1 u u_x - \delta_2 u + \delta_3 u^2 = 0. \quad (10)$$

The parameters δ_1, δ_2 and δ_3 respectively weight the nonlinear advective term the linear growth term and the quadratic saturation term. Following Murray's power law, we set $u = D^p$, where D is the vessel diameter and p the flow diameter exponent, so that the constant solution obtained below can subsequently be interpreted diametrically (Section 4).

Applying the traveling wave transformation $u(x, t) = U(\xi)$, $\xi = x - ct$, Eq.(10) reduces to the ODE

$$cU' + U'' + \delta_1 U U' + \delta_2 U - \delta_3 U^2 = 0. \quad (11)$$

Balancing U'' against U^2 gives $m + 2 = 2m$, i.e. $m = 2$, so the ansatz (4) specializes to

$$U(\xi) = a_0 + a_1(G'/G) + a_2(G'/G)^2 + b_1(F'/F) + b_2(G'/G)(F'/F),$$

$$U(\xi) = a_0 - a_1 F + a_2 F^2 + b_1(F^{-1} - F) - b_2(1 - F^2) \quad (12)$$

Substituting (12) into (11) and collecting the coefficients of each power F^i ($i = 0, \pm 1, \pm 2, \pm 3$) yields nine simultaneous algebraic equations in a_0, a_1, a_2, b_1, b_2 and c (reproduced in full in Appendix A, Eqs. A.1–A.9). Solving this system with Maple yields two admissible parameter families, of which the first is trivial ($a_0 = b_2, a_1 = 0, a_2 = -b_2, b_1 = 0$, leaving the free constant b_2 undetermined by the physics) and the second is the physically meaningful solution

$$a_0 = b_2 + \frac{\delta_2}{\delta_3}, \quad a_1 = 0, \quad a_2 = -b_2, \quad b_1 = 0, \quad b_2 = b_2, \quad c = c. \quad (13)$$

Substitution of (13) into (12) collapses the F -dependence identically, leaving the constant equilibrium traveling-wave solution

$$u_1(x, t) = \frac{\delta_2}{\delta_3} \quad (14)$$

Equation (14) states that, once the reference balance between the linear driving term (δ_2) and the nonlinear saturation term (δ_3) is reached, the model relaxes to a spatially and temporally uniform steady state independent of x and t regardless of the specific value of the wave-speed parameter c or of the free constant b_2 . Figure 1 shows the corresponding surface plot for the representative numerical value $\delta_2/\delta_3 = 14.78$ (Section 4), confirming that $u_1(x, t)$ is flat over the (x, t) -plane.

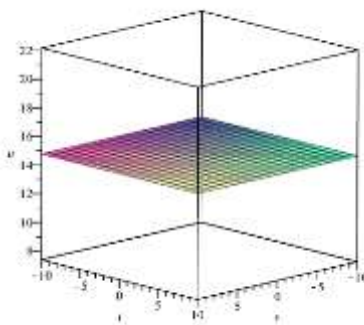


Figure 1. Surface plot of the equilibrium traveling-wave solution $u_1(x, t) = \delta_2/\delta_3 = 14.78$ (Eq. 14), shown here for the original Murray equation; the identical flat surface is obtained for the modified and extended forms (Eqs. 19 and 24).

3.2 The Modified Murray Equation (Cubic Reaction Term)

To test whether the equilibrium solution obtained above is an artifact of the specific quadratic closure $\delta_3 u^2$, we next replace it with a cubic saturation term:

$$u_t - u_{xx} - \delta_1 u u_x - \delta_2 u + \delta_3 u^3 = 0 \quad (15)$$

With $u(x, t) = U(\xi)$, $\xi = x - ct$, Eq. (15) reduces to

$$cU' + U'' + \delta_1 U U' + \delta_2 U - \delta_3 U^3 = 0 \quad (16)$$

Balancing U'' against U^3 now gives $m + 2 = 3m$, i.e. $m = 1$, so the ansatz reduces to

$$U(\xi) = a_0 - a_1 F + a_2 F^2 + b_1(F^{-1} - F) \quad (17)$$

Substituting (17) into (16) and collecting the coefficients of F^i ($i = 0, \pm 1, \dots, 6$) produces a ten-equation algebraic system (Appendix A, Eqs. A.10–A.19). Its non-trivial solution is

$$a_0 = \delta_2/\delta_3, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad c = c \quad (18)$$

which again collapses ansatz (17) to the same constant equilibrium found for the original equation:

$$u_1(x, t) = \delta_2/\delta_3 \quad (19)$$

3.3 The Extended Murray Equation (Quadratic Dissipation Term)

As a second, structurally different modification, we reintroduce the quadratic reaction term of the original equation and add a quadratic gradient (dissipative) term $\delta_4(u_x)^2$, which allows the influence of spatial steepness on the wave behavior to be examined explicitly:

$$u_t - u_{xx} - \delta_1 u u_x - \delta_2 u + \delta_3 u^2 + \delta_4 (u_x)^2 = 0 \quad (20)$$

With the same transformation Eq. (20) reduces to

$$cU' + U'' + \delta_1 U U' + \delta_2 U - \delta_3 U^3 - \delta_4 (U')^2 = 0 \quad (21)$$

Balancing again gives $m = 2$, so the ansatz has the same form as (12):

$$U(\xi) = a_0 - a_1 F + a_2 F^2 + b_1(F^{-1} - F) - b_2(1 - F^2) \quad (22)$$

Collecting the coefficients of F^i ($i = 0, \pm 1, \dots, \pm 4$) now produces an eleven-equation system (Appendix A, Eqs. A.20–A.30), owing to the extra terms generated by $(U')^2$. The non-trivial solution is

$$a_0 = (b_2 \delta_3 + \delta_2)/\delta_3, \quad a_1 = 0, \quad a_2 = -b_2, \quad b_1 = 0, \quad b_2 = b_2, \quad c = c \quad (23)$$

which again reduces to the invariant constant solution

$$u_1(x, t) = \delta_2/\delta_3 \quad (24)$$

3.4 Robustness of the Equilibrium Solution

Equations (14), (19) and (24) show that the Variation of the (G'/G) -expansion method returns the identical non-trivial traveling wave solution $u = \delta_2/\delta_3$, for three structurally distinct versions of Murray's equation: the original quadratic-reaction model, a cubic-reaction modification, and a quadratically dissipative extension. This invariance holds despite the fact that the three models require different balancing orders ($m = 2, 1$ and 2 , respectively) and produce algebraic systems of different size (9, 10 and 11 equations). Because the equilibrium value depends only on the ratio of the linear driving coefficient δ_2 to the nonlinear saturation coefficient δ_3 , and not on the advective coefficient δ_1 , the dissipation coefficient δ_4 , or the wave speed c , we interpret δ_2/δ_3 as an

intrinsic reference state of the underlying physiological system rather than a solution artifact of any one nonlinear closure. This robustness is the mathematical basis for the clinical translation developed in the next section.

4. Clinical Application: Murray's Law and Coronary Artery Diameter Assessment

Murray's power law relates volumetric coronary blood flow Q to vessel diameter D through $Q \propto D^p$. Following Taylor et al. (2022), who used a validated computational fluid dynamics model against invasively measured flow and microvascular resistance in 27 branched human epicardial coronary arteries, the exponent that best reproduces absolute coronary flow is $p = 2.15$, appreciably lower than Murray's original theoretical value of 3.0 and consistent with independent morphometric estimates in the epicardial coronary literature. Adopting the hypothesis $u = D^p$, with $p = 2.15$ (Eq. 2.3.2 of the underlying derivation), the equilibrium solution $u = \delta_2/\delta_3$ obtained in Section 3 for all three forms of Murray's equation can be read as a diameter-derived reference value once numerical values are assigned to δ_2 and δ_3 .

Taking the normal left anterior descending (LAD) artery diameter as $D = 3.5 \text{ mm}$, the reference value is

$$u = D^p = (3.5)^{2.15} = 14.78 \quad (25)$$

and the model parameters can conversely be chosen so that $\delta_2/\delta_3 = 14.78$ reproduces this normal reference state. The relationship is invertible: given any reference value δ_2/δ_3 , the corresponding diameter follows from

$$D = \left(\frac{\delta_2}{\delta_3}\right)^{1/p} \quad (26)$$

For example, $D = (11.38)^{1/2.15} \approx 3.1 \text{ mm}$ and $D = (7.17)^{1/2.15} \approx 2.5 \text{ mm}$. Because δ_2 represents (in the underlying blood-flow model) the resistance experienced by the vessel and δ_3 represents the wall's elastic response to that resistance, the ratio δ_2/δ_3 can be interpreted as an equilibrium index of the balance between resistance and wall compliance an index that, via Eq. (26), maps directly onto a clinically measurable diameter.

Table 1 tabulates this mapping over a clinically relevant range, linking the reference value to a qualitative diagnostic category for the LAD artery, from severe narrowing to severe dilation, with $3.5 \text{ mm} / 14.78$ taken as the normal reference point.

Table 1. Reference values of δ_2/δ_3 mapped onto LAD diameter via $D = (\delta_2/\delta_3)^{1/2.15}$, and the corresponding qualitative diagnostic category.

Reference value (δ_2/δ_3)	LAD diameter (mm)	Diagnostic interpretation
7.17	2.5	Severe narrowing
11.38	3.1	Slight narrowing
14.78	3.5	Normal
20.77	4.1	Slight dilation
25.37	4.5	Severe dilation

Results and Discussion

The central mathematical finding of this study is that the Variation of the (G'/G)-expansion method, applied to three structurally different forms of Murray's nonlinear reaction–diffusion equation, converges in every case on the same non-trivial constant solution, $u = \delta_2/\delta_3$. This is a non-obvious result: the three models differ in balancing order ($m = 2, 1, 2$), in the size of the resulting algebraic system (9, 10, 11 equations) and in which physical mechanism is emphasized — quadratic saturation, cubic saturation, or gradient-driven dissipation — yet the surviving non-trivial equilibrium is identical in all three cases. From a dynamical-systems perspective this is consistent with δ_2/δ_3 being a genuine fixed point of the underlying reaction kinetics: replacing u^2 by u^3 , or adding a term that vanishes identically for spatially uniform u (as $(u_x)^2$ does), cannot move a spatially uniform fixed point, and the method correctly recovers this invariance analytically rather than merely numerically.

Clinically, the translation of this equilibrium into a diameter reference value via Murray's law provides a mathematically explicit if necessarily simplified bridge between a one-dimensional reaction–diffusion description of vessel-wall/flow dynamics and a bedside-interpretable quantity, the LAD diameter. Using the physiologically derived exponent $p = 2.15$ rather than Murray's original cubic exponent is important: had $p = 3.0$ been used, the same reference value of 14.78 would map onto a materially different diameter, illustrating the sensitivity of any

such translation to the exponent chosen and reinforcing the importance of exponent studies such as Taylor et al. (2022) for any prospective clinical use of this framework.

Several limitations should be acknowledged. First, the reaction–diffusion model considered here is one-dimensional and does not capture the branching topology to which Murray's law strictly applies; the diameter mapping in Section 4 should therefore be read as an illustrative reference-value framework rather than a validated diagnostic tool. Second, the algebraic solution families obtained in Section 3 include trivial branches (not reported in detail) alongside the physically meaningful ones; the selection of the non-trivial branch, while standard practice in the (G'/G) -expansion literature, is not itself derived from a stability analysis of the underlying PDE. Third, the exponent $p = 2.15$ was derived from a specific cohort of 27 branched coronary arteries and, as the original authors note, larger and more heterogeneous samples are required before firm exponent values can be adopted for individual patient assessment. Future work should couple the present analytical framework with a linear-stability or numerical-continuation analysis of the constant solution, and should extend the clinical translation to other coronary segments beyond the LAD.

Conclusion

The Variation of the (G'/G) -expansion method was applied to Murray's nonlinear reaction diffusion equation and to two structural modifications a cubic-reaction variant and a quadratic-dissipation extension. In all three cases, the method returns the same physically meaningful equilibrium traveling-wave solution, $u = \delta_2/\delta_3$, demonstrating that this solution is a robust feature of the underlying model rather than an artifact of a particular nonlinear closure. Linking this equilibrium to Murray's power law with the clinically derived exponent $p = 2.15$ yields a reference value of 14.78 that reproduces the normal LAD diameter of 3.5 mm, and a diagnostic look-up table connecting reference values to qualitative categories of coronary narrowing and dilation. Beyond the specific application to Murray's equation, the results illustrate how classical analytical methods for nonlinear PDEs can be combined with contemporary clinical exponent estimates to build simple, interpretable, and testable bridges between mathematical models of vascular physiology and bedside diagnostic reasoning. Extending the framework to branching (rather than one-dimensional) geometries, and subjecting the equilibrium solution to a formal stability analysis, are natural next steps.

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Appendix A. Full Algebraic Systems

The following systems were obtained by substituting the respective ansatz into the reduced ODE and collecting the coefficients of each power of $F(\xi)$; all systems were solved using Maple.

A.1 Original Murray Equation (Eqs. A.1–A.9)

$$F^5: -2a_2^2\delta_1 - 4a_2b_2\delta_1 - 2b_2^2\delta_1 = 0 \quad (\text{A.1})$$

$$F^4: 3a_1a_2\delta_1 + 3a_1b_2\delta_1 - a_2^2\delta_3 + 3a_2b_1\delta_1 - 2a_2b_2\delta_3 + 3b_1b_2\delta_1 - b_2^2\delta_3 + 6a_2 + 6b_2 = 0 \quad (\text{A.2})$$

$$F^3: -2a_0a_2\delta_1 - 2a_0b_2\delta_1 - a_1^2\delta_1 + 2a_1a_2\delta_3 - 2a_1b_1\delta_1 + 2a_1b_2\delta_3 + 2a_2^2\delta_1 + 2a_2b_1\delta_3 + 6a_2b_2\delta_1 - b_1^2\delta_1 + 2b_1b_2\delta_3 + 4b_1^2\delta_1 - 2a_2c - 2b_2c - 2a_1 - 2b_1 = 0 \quad (\text{A.3})$$

$$F^2: a_0a_1\delta_1 - 2a_0a_2\delta_3 + a_0b_1\delta_1 - 2a_0b_2\delta_3 - a_1^2\delta_3 - 3a_1a_2\delta_1 - 2a_1b_1\delta_3 - 4a_1b_2\delta_3 - 4a_2b_1\delta_1 + 2a_2b_2\delta_3 - b_1^2\delta_3 - 5b_1b_2\delta_1 + 2b_2^2\delta_3 + a_1c + a_2\delta_2 + b_1c + b_2\delta_2 - 8a_2 - 8b_2 = 0 \quad (\text{A.4})$$

$$F^1: 2a_0a_1\delta_3 + 2a_0a_2\delta_1 + 2a_0b_1\delta_3 + 2a_0b_2\delta_1 + a_1^2\delta_1 + 2a_1b_1\delta_1 - 2a_1b_2\delta_3 - 2a_2b_1\delta_3 - 2a_2b_2\delta_1 + b_1^2\delta_1 - 4b_1b_2\delta_3 - 2b_2^2\delta_1$$

$$-a_1\delta_2 + 2a_2c - b_1\delta_2 + 2b_2c + 2a_1 + 2b_1 = 0 \quad (\text{A.5})$$

$$\begin{aligned} F^0: & -\delta_3 a_0^2 - \delta_1 a_0 a_1 + 2\delta_3 a_0 b_2 + 2\delta_3 a_1 b_1 + \delta_1 b_2 a_1 + \delta_1 a_2 b_1 \\ & + 2\delta_3 b_1^2 + \delta_1 b_1 b_2 - \delta_3 b_2^2 + \delta_2 a_0 - ca_1 - \delta_2 b_2 + 2a_2 + 2b_2 = 0 \end{aligned} \quad (\text{A.6})$$

$$F^{-1}: -2a_0 b_1 \delta_3 + b_1^2 \delta_1 + 2b_1 b_2 \delta_3 + b_1 \delta_2 - 2b_1 = 0 \quad (\text{A.7})$$

$$F^{-2}: -a_0 b_1 \delta_1 - b_1^2 \delta_3 + b_1 b_2 \delta_1 - b_1 c = 0 \quad (\text{A.8})$$

$$F^{-3}: -b_1^2 \delta_1 + 2b_1 = 0 \quad (\text{A.9})$$

A.2 Modified Murray Equation, Cubic Term (Eqs. A.10–A.19)

$$F^6: -\delta_3 a_2^2 = 0 \quad (\text{A.10})$$

$$F^5: 3a_1 a_2^2 \delta_3 + 3a_2^2 b_1 \delta_3 - 2a_2^2 \delta_1 = 0 \quad (\text{A.11})$$

$$F^4: -3a_0 a_2^2 \delta_3 - 3a_1^2 a_2 \delta_3 - 6a_1 a_2 b_1 \delta_3 - 3a_2 b_1^2 \delta_3 + 3a_1 a_2 \delta_1 + 3a_2 b_1 \delta_1 + 6a_2 = 0 \quad (\text{A.12})$$

$$\begin{aligned} F^3: & 6a_0 a_1 a_2 \delta_3 + 6a_0 a_2 b_1 \delta_3 + a_1^3 \delta_3 + 3a_1^2 b_1 \delta_3 + 3a_1 b_1^2 \delta_3 - 3a_2^2 b_1 \delta_3 \\ & + b_1^3 \delta_3 - 2a_0 a_2 \delta_1 - a_1^2 \delta_1 - 2a_1 b_1 \delta_1 + 2a_2^2 \delta_1 - b_1^2 \delta_1 - 2a_2 c - 2a_1 2b_1 = 0 \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} F^2: & 3a_0^2 a_2 \delta_3 - 3a_0 a_1^2 \delta_3 + 6a_0 a_1 \delta_3 - 3a_0 b_1^2 \delta_3 - 6a_0 a_2 b_1 \delta_3 + 6a_2 b_1^2 \delta_3 + a_0 a_1 \delta_1 + a_0 b_1 \delta_1 - 3a_0 a_2 \delta_1 - \\ & 4a_2 b_1 \delta_1 + a_1 c + a_2 \delta_2 + b_1 c - 8a_2 = 0 \end{aligned} \quad (\text{A.14})$$

$$F^1: 3a_0^2 a_1 \delta_3 + 3a_0^2 b_1 \delta_3 - 6a_1 a_2 b_1 - 3a_1^2 b_1 \delta_3 - 6a_1 b_1^2 \delta_3 + 2a_2 c - b_1 \delta_2 + 2a_1 + 2b_1 = 0 \quad (\text{A.15})$$

$$\begin{aligned} F^0: & -\delta_3 a_0^3 + 6\delta_3 a_0 a_1 b_1 + 6\delta_3 a_0 b_1^2 - 3\delta_3 a_2 b_1^2 - \delta_1 a_0 a_1 + \delta_1 a_2 b_1 \\ & + \delta_2 a_0 - ca_1 + 2a_2 = 0 \end{aligned} \quad (\text{A.16})$$

$$F^{-1}: -3a_0^2 b_1 \delta_3 + 3a_1 b_1^2 \delta_3 + 3b_1^3 \delta_3 + b_1^2 \delta_1 + b_1 \delta_2 - 2b_1 = 0 \quad (\text{A.17})$$

$$F^{-2}: -3a_0^2 b_1 \delta_3 - a_0 b_1 \delta_1 - b_1 c = 0 \quad (\text{A.18})$$

$$F^{-3}: -b_1^3 - \delta_3 b_1^2 \delta_1 + 2b_1 = 0 \quad (\text{A.19})$$

A.3 Extended Murray Equation, Dissipation Term (Eqs. A.20–A.30)

$$F^6: -4a_2^2 - 8a_2 b_2 \delta_4 - 4b_2^2 \delta_4 = 0 \quad (\text{A.20})$$

$$F^5: 4a_1 a_2 \delta_4 + 4a_1 b_2 \delta_4 - 2a_2^2 \delta_2 + 4a_2 b_1 \delta_4 - 4a_2 b_2 \delta_1 + 4b_1 b_2 \delta_4 - 2b_2^2 \delta_1 = 0 \quad (\text{A.21})$$

$$\begin{aligned} F^4: & -a_1^2 \delta_4 + 3a_1 a_2 \delta_1 a - 2a_1 b_1 \delta_4 + 3a_1 b_2 \delta_1 - a_2^2 \delta_3 + 8a_2^2 \delta_4 + 3a_2 b_1 \delta_1 - 2a_2 b_2 \delta_3 + 16a_2 b_2 \delta_4 - b_1^2 + \\ & 3b_1 b_2 \delta_1 - b_2^2 \delta_3 + 8b_2^2 \delta_4 + 6a_2 + 6b_2 = 0 \end{aligned} \quad (\text{A.22})$$

$$\begin{aligned} F^3: & -2a_0 a_2 \delta_1 - 2a_0 b_2 \delta_1 - a_1^2 \delta_1 + 2a_1 a_2 \delta_3 - 8a_1 a_2 \delta_4 - 2a_1 b_1 \delta_1 + 2a_1 b_2 \delta_3 - 8a_1 b_2 \delta_4 + 2a_2^2 \delta_1 + \\ & 2a_2 b_1 \delta_3 - 4a_2 b_1 \delta_4 + 6a_2 b_2 \delta_1 - b_1^2 \delta_1 + 2b_1 b_2 \delta_3 - 4b_1 b_2 \delta_4 + 4b_2^2 \delta_1 - 2a_2 c - 2b_2 c - 2a_1 - 2b_1 = 0 \end{aligned} \quad (\text{A.23})$$

$$\begin{aligned} F^2: & a_0 a_1 \delta_1 - 2a_0 a_2 \delta_3 + a_0 b_1 \delta_1 - 2a_0 b_2 \delta_3 - a_2^2 \delta_3 + 2a_2^2 \delta_4 - 3a_1 a_2 \delta_1 - 2a_1 b_1 \delta_3 + 2a_1 b_1 \delta_4 - 4a_1 b_2 \delta_1 - \\ & 4a_2^2 \delta_4 - 4a_2 b_1 \delta_1 + 2a_1 b_2 \delta_3 - 8a_2 b_2 \delta_4 - b_1^2 \delta_3 - 5b_1 b_2 \delta_1 + 2b_2^2 \delta_3 - 4b_1^2 \delta_4 + a_1 c + a_2 \delta_2 + b_1 c + b_2 \delta_2 - \\ & 8a_2 - 8b_2 = 0 \end{aligned} \quad (\text{A.24})$$

$$\begin{aligned} F^1: & 2a_0 a_1 \delta_3 + 2a_0 a_2 \delta_1 + 2a_0 b_1 \delta_3 + 2a_0 b_2 \delta_1 + a_2^2 \delta_1 + 4a_1 a_2 \delta_4 + 2a_1 b_1 \delta_1 - 2a_1 b_2 \delta_3 + 4a_1 b_2 \delta_4 - \\ & 2a_2 b_1 \delta_3 - 4a_2 b_1 \delta_4 - 2a_2 b_2 \delta_1 + b_2^2 \delta_1 - 4b_1 b_2 \delta_3 - 4b_1 b_2 \delta_4 - 2b_2^2 \delta_1 - a_1 \delta_2 + 2a_2 c - b_1 \delta_2 + 2b_2 c + \\ & 2a_1 + 2b_1 = 0 \end{aligned} \quad (\text{A.25})$$

$$\begin{aligned} F^0: & -a_2^2 \delta_3 - a_0 a_1 \delta_1 + 2a_0 b_2 \delta_3 - a_1^2 \delta_4 + 2a_1 b_1 \delta_3 + 2a_1 b_1 \delta_4 + a_1 b_2 \delta_1 + a_2 b_1 \delta_1 + 2b_1^2 \delta_3 + 2b_1^2 \delta_4 + \\ & b_1 b_2 \delta_1 - b_2^2 \delta_3 + a_0 \delta_2 - a_1 c - b_2 \delta_2 + 2a_2 + 2b_2 = 0 \end{aligned} \quad (\text{A.26})$$

$$F^{-1}: -2a_0 b_1 \delta_3 + 4a_2 b_1 \delta_4 + b_1^2 \delta_1 + 2b_1 b_2 \delta_3 + 4b_1 b_2 \delta_4 + b_1 \delta_2 - 2b_1 = 0 \quad (\text{A.27})$$

$$F^{-2}: -a_0 b_1 \delta_1 - 2a_1 b_1 \delta_4 - b_1^2 \delta_3 + b_1 b_2 \delta_1 - b_1 c = 0 \quad (\text{A.28})$$

$$F^{-3}: -b_1^2 \delta_1 + 2b_1 = 0 \quad (\text{A.29})$$

$$F^{-4}: -\delta_4 b_1^2 = 0 \quad (A.30)$$

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