



Spectral Analysis of the Generalized difference operator $B(r, s)$ and its Adjoint on a sequence space of convergent Series

Suad H. Abu-Janah*

Department of Mathematics, Faculty of Sciences, University of Elmergib, Msallata, Libya

التحليل الطيفي لمؤثر الفرق المعمم $B(r, s)$ ومؤثره المرافق على فضاء متتاليات لمتسلسلات متقاربة

سعاد هويدى أبو جناح*

قسم الرياضيات، كلية العلوم، جامعة المرقب، مسلاتة، ليبيا

*Corresponding author: shabujanah@elmergib.edu.ly

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Abstract:

This research is investigates the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence space cs . The main purpose of this work is to provide a significant improvement of the spectral results previously published in the literature. Specifically, we demonstrate that the results established by Durn, Yildirim and Kilic in [5] and Dutta and Tripathy in [6] regarding the spectrum and its subdivisions of the operator $B(r, s)$ on cs contain inconsistencies. In this study, we provide updated proofs and a more precise determination of the fine spectrum, including the residual, continuous and compression spectra. In particular, the point spectrum of the adjoint operator $B(r, s)^*$ is reanalyzed using the correct dual space $cs^* \cong bv$, rather than the previously assumed identification $cs^* \cong \ell_1$. This correction leads to new descriptions of the point spectrum $B(r, s)^*$ and consequently to a revised determination of the corresponding, continuous and compression spectra of $B(r, s)$. These results contribute to addressing some gaps in previous works and improving the corresponding results, ensuring greater accuracy in extended studies of these results and providing a basis for further studies involving examples and applications in different fields, given the importance of spectral analysis in many modern applications.

Keywords: Spectrum, Infinite Matrices, Sequence Spaces.

الملخص

يتناول هذا البحث دراسة الطيف الدقيق لمؤثر الفرق المعمم $B(r, s)$ على فضاء المتتاليات cs . الغرض الرئيسي من هذا العمل هو تقديم تحسين للنتائج الطيفية التي نشرت سابقاً في المراجع العلمية [6,5]. وبشكل محدد، نبرهن أن النتائج التي أثبتت في [6,5] فيما يتعلق بالطيف وتقسيماته للمؤثر $B(r, s)$ على الفضاء cs تحتوي على تناقضات. في هذه الدراسة، نقدم براهين محدثة ونتائج أكثر دقة لبعض الأطياف، بما في ذلك الطيف النقطي للمؤثر المرافق، الطيف المتبقي، الطيف المستمر والضغوط. وعلى وجه الخصوص، أعيد تحليل الطيف النقطي للمؤثر المرافق $B(r, s)^*$ باستخدام الفضاء المرافق الصحيح $cs^* \cong bv$ ، بدلاً من التعيين المفترض سابقاً $cs^* \cong \ell_1$. وقد أدى هذا التصحيح إلى توصيفات جديدة للطيف النقطي للمؤثر $B(r, s)^*$ ، وبالتالي إلى إعادة تحديد الأجزاء المقابلة من الطيف المتبقي والمتصل والضغوط للمؤثر $B(r, s)$. وتساهم هذه النتائج في معالجة بعض الثغرات الموجودة في الأعمال السابقة

وتحسين النتائج المناظرة، بما يضمن دقة أكبر في الدراسات الممتدة لهذه النتائج، ويمهد لدراسات تتضمن أمثلة وتطبيقات في مجالات مختلفة، نظرًا لأهمية التحليل الطيفي في الكثير من التطبيقات الحديثة.

الكلمات المفتاحية: الطيف، المصفوفات اللانهائية، فضاءات المتتاليات.

Introduction

We begin our discussion by recalling the fundamental definitions and notations required throughout this work.

Definition 1. [12, 15] The Banach spaces bv of all sequences of bounded variation and cs of convergent series are defined by:

$$bv = \left\{ x = (x_k)_{k=0}^{\infty} : x_k \in \mathbb{C}, \sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty \right\},$$

$$cs = \left\{ x = (x_k)_{k=0}^{\infty} : x_k \in \mathbb{C}, \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^n x_k \right| < \infty \right\},$$

with norms $\|x\|_{bv} = \sum_{k=0}^{\infty} |x_{k+1} - x_k|$ and $\|x\|_{cs} = \sup_{n \in \mathbb{N}} |\sum_{k=0}^n x_k|$.

Definition 2. Let the spectrum of a bounded linear operator T defined on a complex infinite dimensional Banach space X be denoted by $\sigma(T, X)$ as the set of complex numbers $\lambda \in \mathbb{C}$ for which the operator $(T - \lambda I)$ is not invertible, where I is the identity mapping of X onto itself.

$$\sigma(T, X) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

If $(T - \lambda I)$ has an inverse which is linear, we call it the resolvent operator of T .

The resolvent set $\rho(T, X) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is a bijection}\}$,

The set $\sigma(T, X)$ is the complement of the set $\rho(T, X)$ in the complex plane $\mathbb{C}$.

. The spectrum is partition into three main disjoint subsets whose union gives the entire spectrum, serving as the primary classification:

Point spectrum $\sigma_p(T, X)$: The set of eigenvalues.

Continuous spectrum $\sigma_c(T, X)$: The set where the operator is not invertible, yet its range is dense.

Residual spectrum $\sigma_r(T, X)$: The set where the operator is not invertible and its range is not dense.

A secondary classification focuses on the approximate behavior and the range properties of the operator, comprising:

Approximate point spectrum $\sigma_{ap}(T, X)$: The set capturing approximate invertibility.

Compression spectrum $\sigma_{co}(T, X)$: The set where the range of the operator fails to be dense.

Defect spectrum $\sigma_{\delta}(T, X)$: The set associated with the deficiency or lack of surjectivity.

Furthermore, the spectrum is categorized into types I, II and III based on specific range behavior and invertibility criteria. For example, the subset consisting of those λ for which the state of the operator $T - \lambda I$ is III_1 will be denoted by $III_1(T, X)$. For detailed information regarding these spectral classifications, see [11,13,14].

These definitions and applying Proposition 1.3(e) in [2, p.28] naturally give rise to the following relations:

$$\begin{aligned}
\sigma(T, X) &= \sigma_p(T, X) \cup \sigma_r(T, X) \cup \sigma_c(T, X), \\
\sigma_{ap}(T, X) &= \sigma(T, X) \setminus III_1(T, X), \\
\sigma_\delta(T, X) &= \sigma(T, X) \setminus I_3(T, X), \\
\sigma_{co}(T, X) &= \sigma_p(T^*, X^*).
\end{aligned} \tag{1.1}$$

Lemma 1. [14, Page 60]. T has a bounded inverse if and only if T^* is onto.

Lemma 2. [9, pp. 20; 10, pp. 38]. If T is a linear operator on a complex normed space X into itself, then $III_1(T, X)$ is an open set.

2. The main results

The generalized difference operator $B(r, s)$ is defined on a Banach Sequence space μ as

$$B(r, s)x = (rx_0, rx_1 + sx_0, rx_2 + sx_1, \dots), \quad x = (x_k)_{k=0}^\infty \in \mu,$$

Where $r, s \in \mathbb{R}$, $s \neq 0$.

This operator can be represented by a band matrix as:

$$B(r, s) = \begin{bmatrix} r & 0 & 0 & \cdots \\ s & r & 0 & \cdots \\ 0 & s & r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (s \neq 0).$$

The fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c have been studied by Altay and Başar [1]. Also, the spectrum and fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_1 and bv have been studied by Furkan et al. [8]. The fine spectrum of the operator $B(r, s)$ over ℓ_p and bv_p where $1 < p < \infty$ has been examined by Bilgiç and Furkan [4]. Dutta and Tripathy [6] have studied the fine spectrum of the generalized difference operator $B(r, s)$ over the space of convergent series. Recently, the fine spectrum of the operator $B(r, s)$ over bv_0 has been studied by El-Shabrawy and Abu-Janah [7]. Moreover, some additional results concerning on other well-known classification of the spectrum of $B(r, s)$ on cs have been given by Durna, Yildirim and Kilic in [5], also by Başar on other space in [3].

Lemma 3 [15, Example 6B, Page 130]. The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(cs)$ if and only if

1. $\sup_k \sum_{n=1}^m |a_{nk} - a_{n,k-1}| < \infty$,
2. $\sum_{n=1}^\infty a_{nk}$ is convergent for each k .

The boundedness of the operator $B(r, s)$ on cs has been proven in [6]. Here, we present another proof.

Theorem 2.1. The operator $B(r, s): cs \rightarrow cs$ is a bounded linear operator.

Proof. It is clear that the operator $B(r, s)$ on the sequence space cs is linear.

So, we shall prove that the conditions of Lemma 3 are satisfied; thereby $B(r, s)$ is a matrix transformation on cs into itself. Let

$$A_m = \sum_{k=1}^{\infty} \left| \sum_{n=1}^m (b_{nk} - b_{n,k-1}) \right|.$$

Then

$$\begin{aligned} A_1 &= \sum_{k=1}^{\infty} \left| \sum_{n=1}^2 (b_{nk} - b_{n,k-1}) \right| \\ &= |b_{11} - b_{10}| + |b_{12} - b_{11}| + |b_{13} - b_{12}| + |b_{14} - b_{13}| + \cdots \\ &= 2|r|, \end{aligned}$$

$$\begin{aligned} A_2 &= \sum_{k=1}^{\infty} \left| \sum_{n=1}^3 (b_{nk} - b_{n,k-1}) \right| \\ &= |b_{11} - b_{10} + b_{21} - b_{20}| + |b_{12} - b_{11} + b_{22} - b_{21}| \\ &\quad + |b_{13} - b_{12} + b_{23} - b_{22}| + |b_{14} - b_{13} + b_{24} - b_{23}| + \cdots \\ &= |r + s| + |s| + |r|, \end{aligned}$$

$$\begin{aligned} A_3 &= \sum_{k=1}^{\infty} \left| \sum_{n=1}^4 (b_{nk} - b_{n,k-1}) \right| \\ &= |b_{11} - b_{10} + b_{21} - b_{20} + b_{31} - b_{30}| + |b_{12} - b_{11} + b_{22} - b_{21} + b_{32} - b_{31}| \\ &\quad + |b_{13} - b_{12} + b_{23} - b_{22} + b_{33} - b_{32}| + |b_{14} - b_{13} + b_{24} - b_{23} + b_{34} - b_{33}| \\ &\quad + \cdots \\ &= |r + s| + |s| + |r|. \end{aligned}$$

Then, we can obtain that

$$A_m = \sum_{k=1}^{\infty} \left| \sum_{n=1}^m (b_{nk} - b_{n,k-1}) \right| = \begin{cases} 2|r| & \text{if } m = 1 \\ |r + s| + |s| + |r| & \text{if } m \geq 2. \end{cases}$$

Thus, for all $m \geq 1$

$$\sup_m A_m = \sum_{k=1}^{\infty} \left| \sum_{n=1}^m (b_{nk} - b_{n,k-1}) \right| = |r + s| + |s| + |r| < \infty.$$

Additionally, one can prove that

$$\sum_{n=1}^{\infty} a_{nk} = r + s, \quad k > 1.$$

So $\sum_{n=1}^{\infty} a_{nk}$ is convergent for each $k > 1$. ■

Due to a clear error in the previous two works [5, 6] regarding the point spectrum of the adjoint operator of $B(r, s)$, which affected some later results, we present a precise correction for this spectrum and its proof to build better results. We also provide another proof for the spectral theorem of the operator $B(r, s)$ on cs , which was previously established in earlier work [6].

Theorem 2.2. $\sigma(B(r, s), cs) = \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}$.

Proof. Suppose $\lambda \notin \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}$. Then $|\lambda - r| > |s|$. So, the matrix $B(r, s) - \lambda I$ is triangle, and so has an inverse, which can be formally given by the matrix

$$(c_{nk}) = \begin{bmatrix} \frac{1}{r-\lambda} & 0 & 0 & 0 & 0 & \cdots \\ -s & \frac{1}{r-\lambda} & 0 & 0 & 0 & \cdots \\ \frac{s^2}{(r-\lambda)^2} & \frac{-s}{r-\lambda} & \frac{1}{r-\lambda} & 0 & 0 & \cdots \\ \frac{-s^3}{(r-\lambda)^3} & \frac{s^2}{(r-\lambda)^2} & \frac{-s}{r-\lambda} & \frac{1}{r-\lambda} & 0 & \cdots \\ \frac{s^4}{(r-\lambda)^4} & \frac{-s^3}{(r-\lambda)^3} & \frac{s^2}{(r-\lambda)^2} & \frac{-s}{r-\lambda} & \frac{1}{r-\lambda} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let

$$B_m = \sum_{k=1}^{\infty} \left| \sum_{n=1}^m (c_{nk} - c_{n,k-1}) \right|,$$

so

$$\begin{aligned} B_1 &= \sum_{k=1}^{\infty} \left| \sum_{n=1}^1 (c_{nk} - c_{n,k-1}) \right| \\ &= 2 \left| \frac{1}{r-\lambda} \right|, \\ B_2 &= \sum_{k=1}^{\infty} \left| \sum_{n=1}^2 (c_{nk} - c_{n,k-1}) \right| \\ &= \left| \frac{1}{r-\lambda} - \frac{s}{(r-\lambda)^2} \right| + \left| \frac{s}{(r-\lambda)^2} \right| + \left| \frac{1}{r-\lambda} \right|, \\ B_3 &= \sum_{k=1}^{\infty} \left| \sum_{n=1}^3 (c_{nk} - c_{n,k-1}) \right| \\ &= \left| \frac{1}{r-\lambda} - \frac{s}{(r-\lambda)^2} + \frac{s^2}{(r-\lambda)^3} \right| + \left| \frac{s^2}{(r-\lambda)^3} \right| + \left| \frac{s}{(r-\lambda)^2} \right| + \left| \frac{1}{r-\lambda} \right|. \end{aligned}$$

Then, we have

$$\begin{aligned} B_m &= \left| \frac{1}{r-\lambda} \right| \left[\left| 1 - \frac{s}{r-\lambda} + \frac{s^2}{(r-\lambda)^2} - \frac{s^3}{(r-\lambda)^3} + \cdots + \frac{(-1)^{m-1} s^{m-1}}{(r-\lambda)^{m-1}} \right| + \left| \frac{s^{m-1}}{(r-\lambda)^{m-1}} \right| \right. \\ &\quad \left. + \left| \frac{s^{m-2}}{(r-\lambda)^{m-2}} \right| + \cdots + \left| \frac{s}{r-\lambda} \right| + 1 \right] \\ &= \left| \frac{1}{r-\lambda} \right| \left[\left| \sum_{i=0}^{m-1} \left(\frac{-s}{r-\lambda} \right)^i \right| + \sum_{i=0}^{m-1} \left| \frac{s}{r-\lambda} \right|^i \right] \end{aligned}$$

$$\begin{aligned}
&\leq \left| \frac{1}{r-\lambda} \right| \left[\sum_{i=0}^{m-1} \left| \frac{s}{r-\lambda} \right|^i + \sum_{i=0}^{m-1} \left| \frac{s}{r-\lambda} \right|^i \right] \\
&= \left| \frac{2}{r-\lambda} \right| \sum_{i=0}^{m-1} \left| \frac{s}{r-\lambda} \right|^i, \quad m \geq 2.
\end{aligned}$$

Then, by induction we obtain

$$B_m \leq \left| \frac{2}{r-\lambda} \right| \sum_{i=0}^{\infty} \left| \frac{s}{r-\lambda} \right|^i.$$

Since $\sum_{i=0}^{\infty} \left| \frac{s}{r-\lambda} \right|^i$ is convergent for $\left| \frac{s}{r-\lambda} \right| < 1$, then $\sup_m B_m < \infty$ for $|r - \lambda| > |s|$. Additionally, for each fixed k ,

$$\begin{aligned}
\sum_{n=1}^{\infty} a_{nk} &= \frac{1}{r-\lambda} + \frac{-s}{(r-\lambda)^2} + \frac{s^2}{(r-\lambda)^3} + \frac{-s^3}{(r-\lambda)^4} + \cdots \\
&= \frac{1}{r-\lambda} \left[1 + \frac{-s}{r-\lambda} + \frac{s^2}{(r-\lambda)^2} + \frac{-s^3}{(r-\lambda)^3} + \cdots \right] \\
&= \frac{1}{r-\lambda} \sum_{k=0}^{\infty} \left(\frac{-s}{r-\lambda} \right)^k,
\end{aligned}$$

which is convergent for $|s| < |r - \lambda|$.

Then, from Lemma 3, $(B(r, s) - \lambda I)^{-1}: cs \rightarrow cs$ is bounded. So, $\lambda \notin \sigma(B(r, s), cs)$. Hence $\sigma(B(r, s), cs) \subseteq \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}$.

Conversely, if $\lambda \notin \sigma(B(r, s), cs)$, then similar arguments in the proof of Theorem

2.1 in [7], show that $\{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\} \subseteq \sigma(B(r, s), cs)$. ■

In reference [6], the results concerning the operator's and point spectrum are correct and match our results. However, the theorem used to deduce the spectrum of the adjoint operator of $B(r, s)$ on cs^* is inaccurate, which affects the derivation of the residual and continuous spectrum. Here, we present the necessary revisions and improvements.

Theorem 2.3. The following statements hold:

1. $\sigma_p(B(r, s), cs) = \emptyset$,
2. $\sigma_p(B(r, s)^*, cs^*) = \{\lambda \in \mathbb{C}: |\lambda - r| < |s|\} \cup \{r + s\}$,
3. $\sigma_r(B(r, s), cs) = \{\lambda \in \mathbb{C}: |\lambda - r| < |s|\} \cup \{r + s\}$,
4. $\sigma_c(B(r, s), cs) = \{\lambda \in \mathbb{C}: |\lambda - r| = |s|\} \setminus \{r + s\}$.

Proof. In [6], the proof depends on an assumption that $cs^* \cong \ell_1$ but $cs^* \cong bv$ [15].

So, we only give the proof of statement (2).

Suppose that $B(r, s)^* f = \lambda f$ for $f = \{f_k\}_{k=0}^{\infty} \neq \theta$ in $cs^* \cong bv$. Then, we obtain

$$f_k = \left(\frac{\lambda - r}{s} \right)^k f_0, \quad k \geq 1.$$

Clearly, $r \in \sigma_p(B(r, s)^*, cs^*)$. On the other hand, if $\lambda \neq r$, then $f_k \neq 0$ for all $k \in \mathbb{N}$ and

$$\begin{aligned} \sum_{k=0}^{\infty} |f_{k+1} - f_k| &= \sum_{k=0}^n \left| \left(\frac{\lambda - r}{s} \right) f_k - f_k \right| \\ &= \sum_{k=0}^{\infty} \left| \left(\frac{\lambda - r}{s} \right) - 1 \right| |f_k| \\ &= \sum_{k=0}^{\infty} \left| \left(\frac{\lambda - r}{s} \right) - 1 \right| |f_0| \left| \frac{\lambda - r}{s} \right|^k \\ &= \left| \left(\frac{\lambda - r}{s} \right) - 1 \right| |f_0| \sum_{k=0}^{\infty} \left| \frac{\lambda - r}{s} \right|^k. \end{aligned}$$

Since $f = (f_k)_{k=0}^{\infty} \in bv$, so $\sum_{k=0}^{\infty} |f_{k+1} - f_k| < \infty$ if and only if $|\lambda - r| < s$ or $\lambda = \{r + s\}$. Hence, $\sigma_p(B(r, s)^*, cs^*) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\} \cup \{r + s\}$.

Consequently, results (3) and (4) are obtained from the following relations:

$$\sigma_r(B(r, s), cs) = \sigma_p(B(r, s)^*, cs^*) \setminus \sigma_p(B(r, s), cs),$$

$$\sigma_c(B(r, s), cs) = \sigma(B(r, s), cs) \setminus [\sigma_p(B(r, s), cs) \cup \sigma_r(B(r, s), cs)]. \blacksquare$$

For the third classification of the spectrum in this work, we will determine solely $III_1(B(r, s), cs)$ throughout the following theorem, whose proof follows the same steps as a similar theorem in our work [7]. The remaining subdivisions are easily obtained from the preceding results.

Theorem 2.4.

$$III_1(B(r, s), cs) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}.$$

Proof. Let $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$. Similar arguments in the proof of Theorem 2.3 in [7] implies that $\overline{R(B(r, s) - \lambda I)} \neq cs$ and $B(r, s) - \lambda I$ has an inverse. Next, to show that $(B(r, s) - \lambda I)^{-1}$ is bounded, it is enough to show by Lemma 1 that $B(r, s)^* - \lambda I$ is onto. Given $y = (y_k)_{k=0}^{\infty} \in cs^* \cong bv$, we must find $x = (x_k)_{k=0}^{\infty} \in bv$ such that $(B(r, s)^* - \lambda I)x = y$ and direct calculations show that $(r - \lambda)x_k + sx_{k+1} = y_k$, for all $k \in \mathbb{N}$. Then, we have

$$x_1 = \frac{1}{s}y_0 + \frac{(\lambda - r)}{s}x_0,$$

$$x_2 = \frac{1}{s}y_1 + \frac{(\lambda - r)}{s^2}y_0 + \frac{(\lambda - r)^2}{s^2}x_0,$$

$\vdots$

$$x_k = \frac{1}{s}y_{k-1} + \frac{(\lambda - r)}{s^2}y_{k-2} + \cdots + \frac{(\lambda - r)^{k-2}}{s^{k-1}}y_1 + \frac{(\lambda - r)^{k-1}}{s^k}y_0 + \frac{(\lambda - r)^k}{s^k}x_0,$$

So,

$$\begin{aligned}
& \sum_{k=0}^n |x_{k+1} - x_k| = \\
& = \left| \frac{1}{s} y_0 + \frac{(\lambda - r)}{s} x_0 - x_0 \right| + \left| \frac{1}{s} y_1 + \frac{(\lambda - r)}{s^2} y_0 + \frac{(\lambda - r)^2}{s^2} x_0 - \frac{1}{s} y_0 - \frac{(\lambda - r)}{s} x_0 \right| \\
& + \left| \frac{1}{s} y_2 + \frac{(\lambda - r)}{s^2} y_1 + \frac{(\lambda - r)^2}{s^2} y_0 + \frac{(\lambda - r)^3}{s^3} x_0 - \frac{1}{s} y_1 - \frac{(\lambda - r)}{s^2} y_0 - \frac{(\lambda - r)^2}{s^2} x_0 \right| \\
& + \dots + \left| \frac{1}{s} y_n + \frac{(\lambda - r)}{s^2} y_{n-1} + \dots + \frac{(\lambda - r)^{n-2}}{s^{n-1}} y_2 + \frac{(\lambda - r)^{n-1}}{s^n} y_1 + \frac{(\lambda - r)^n}{s^{n+1}} y_0 \right. \\
& + \frac{(\lambda - r)^{n+1}}{s^{n+1}} x_0 - \frac{1}{s} y_{n-1} - \frac{(\lambda - r)}{s^2} y_{n-2} - \dots - \frac{(\lambda - r)^{n-3}}{s^{n-2}} y_2 - \frac{(\lambda - r)^{n-2}}{s^{n-1}} y_1 \\
& \left. - \frac{(\lambda - r)^{n-1}}{s^n} y_0 + \frac{(\lambda - r)^n}{s^n} x_0 \right| \\
& \leq \frac{1}{|s|} |y_0| + \left| \frac{\lambda - r}{s} \right| |x_0| + |x_0| + \frac{1}{|s|} |y_1 - y_0| + \frac{1}{|s|} |y_0| \left| \frac{\lambda - r}{s} \right| + \left| \frac{\lambda - r}{s} \right|^2 |x_0| \\
& + \left| \frac{\lambda - r}{s} \right| |x_0| + \frac{1}{|s|} |y_2 - y_1| + \frac{1}{|s|} |y_1 - y_0| \left| \frac{\lambda - r}{s} \right| + \frac{1}{|s|} |y_0| \left| \frac{\lambda - r}{s} \right|^2 + \left| \frac{\lambda - r}{s} \right|^3 |x_0| \\
& + \left| \frac{\lambda - r}{s} \right|^2 |x_0| + \dots + \frac{1}{|s|} |y_n - y_{n-1}| + \frac{1}{|s|} |y_{n-1} - y_{n-2}| \left| \frac{\lambda - r}{s} \right| + \dots \\
& + \frac{1}{|s|} |y_2 - y_1| \left| \frac{\lambda - r}{s} \right|^{n-2} + \frac{1}{|s|} |y_1 - y_0| \left| \frac{\lambda - r}{s} \right|^{n-1} + \frac{1}{|s|} |y_0| \left| \frac{\lambda - r}{s} \right|^n \\
& + \left| \frac{\lambda - r}{s} \right|^{n+1} |x_0| + \left| \frac{\lambda - r}{s} \right|^n |x_0| \\
& = \frac{1}{|s|} (|y_0| + |y_1 - y_0| + |y_2 - y_1| + \dots + |y_n - y_{n-1}|) \\
& + \frac{1}{|s|} (|y_0| + |y_1 - y_0| + |y_2 - y_1| + \dots + |y_{n-1} - y_{n-2}|) \left| \frac{\lambda - r}{s} \right| \\
& + \frac{1}{|s|} (|y_0| + |y_1 - y_0| + |y_2 - y_1| + \dots + |y_{n-2} - y_{n-3}|) \left| \frac{\lambda - r}{s} \right|^2 \\
& + \frac{1}{|s|} (|y_0| + |y_1 - y_0| + |y_2 - y_1| + \dots + |y_{n-3} - y_{n-4}|) \left| \frac{\lambda - r}{s} \right|^3 + \dots \\
& + \frac{1}{|s|} (|y_0| + |y_1 - y_0| + |y_2 - y_1|) \left| \frac{\lambda - r}{s} \right|^{n-2} + \frac{1}{|s|} (|y_0| + |y_1 - y_0|) \left| \frac{\lambda - r}{s} \right|^{n-1} \\
& + \frac{1}{|s|} |y_0| \left| \frac{\lambda - r}{s} \right|^n
\end{aligned}$$

$$+ \left(1 + 2 \left| \frac{\lambda - r}{s} \right| + 2 \left| \frac{\lambda - r}{s} \right|^2 + 2 \left| \frac{\lambda - r}{s} \right|^3 + \dots + 2 \left| \frac{\lambda - r}{s} \right|^n + 2 \left| \frac{\lambda - r}{s} \right|^{n+1} \right) |x_0|$$

$$\leq \frac{1}{|s|} \sum_{k=0}^n |y_k - y_{k-1}| \left(\sum_{k=0}^n \left| \frac{\lambda - r}{s} \right|^k \right) + \left(2 \sum_{k=0}^{n+1} \left| \frac{\lambda - r}{s} \right|^k \right) |x_0|.$$

Since $|\lambda - r| < |s|$, then $\sum_{k=0}^{\infty} \left| \frac{\lambda - r}{s} \right|^k < \infty$, and so $\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty$, where $\sum_{k=0}^{\infty} |y_k - y_{k-1}| < \infty$.

Therefore $x = (x_k) \in bv$. Thus $B(r, s)^* - \lambda I$ is onto.

Then $\{\lambda \in \mathbb{C} : |\lambda - r| < |s|\} \subseteq III_1(B(r, s), cs)$. Also, by Lemma 2 we have

$$III_1(B(r, s), cs) \subseteq \text{int}(\{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}. \blacksquare$$

The results derived in Reference [5] are an extension of those established in Reference [6], which studied the operator's spectrum and its main subdivisions. As a result, some inaccuracies also appeared in certain results of Reference [5].

The following theorem shows how the value of the operator's compression spectrum in Reference [5] differs from the corresponding value in our results, as we improved this result to arrive at the correct and accurate value.

Theorem 2.5. The following statements hold:

1. $\sigma_{ap}(B(r, s), cs) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\},$
2. $\sigma_{co}(B(r, s), cs) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\} \cup \{r + s\},$
3. $\sigma_{\delta}(B(r, s), cs) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}.$

By using relations given in Eq. (1.1), the proof follows immediately.

" In [5], $\sigma_{co}(B(r, s), cs) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$ ".

Conclusion

The results obtained in this study provide corrections to previous spectral results for the operator $B(r, s)$ on the sequence spaces cs . Such corrections are essential to ensure that future research relies on accurate mathematical results and to prevent the propagation of errors. Looking ahead, the operator $B(r, s)$ can be generalized to a triple-band operator $T(r, s, q)$, which will be the focus of our next work. This generalization is important because the introduction of a third band provides a broader operator structure and allows the spectral properties to be investigated under more general interactions between the coefficients. Thus, the study of $T(r, s, q)$ will provide a natural extension of the spectral analysis of $B(r, s)$ and may identify spectral properties that cannot be described by the two-band structure alone.

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