



Non-Graded Ideals and Gorenstein Quotient Rings

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المُثَل غير المُصَنَّفَة وحلقات حاصل قسمة غورنشتاين

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Abstract:

This paper explores the existence and classification of non-graded ideals in commutative rings, focusing on those whose quotient rings are Gorenstein. While previous works have thoroughly examined graded ideals with Gorenstein quotients, the non-graded case presents unique algebraic challenges and phenomena. We investigate conditions under which non-graded ideals yield Gorenstein quotient rings, and discuss their structural properties. Our results indicate that, under specific algebraic constraints, the collection of non-graded ideals can, depending on the characteristics of the ambient ring.

Keywords: Non-graded ideals, Gorenstein rings, Algebraic geometry, Homological algebra.

المُلخَص

تستكشف هذه الورقة البحثية وجود وتصنيف المُثَل غير المُدرَّجَة في الحلقات التبادلية، مع التركيز على تلك التي تكون حلقات قسمتها من نوع غورنشتاين. في حين أن الدراسات السابقة قد فحصت بدقة المُثَل المُدرَّجَة ذات حلقات القسمة من نوع غورنشتاين، فإن الحالة غير المُدرَّجَة تُقدِّم تحديات وظواهر جبرية فريدة. نُثبت شروطاً ضرورية وكافية لوجود هذه المُثَل، ونُحلِّل كمياتها الكامنة، ونناقش خصائصها البنيوية. تُشير نتائجنا إلى أنه، في ظل قيود جبرية مُحدَّدة، يُمكن أن تكون مجموعة المُثَل غير المُدرَّجَة محدودة أو غير محدودة أو تافهة، وذلك اعتماداً على خصائص الحلقة المحيطة.

الكلمات المفتاحية: المُثَل غير المُدرَّجَة، حلقات غورنشتاين، الهندسة الجبرية، الجبر التماثلي

Introduction

The study of ideals and quotient rings plays a vital role in understanding the structural and homological properties of algebraic systems. Among these, non-graded ideals occupy a significant position, especially in contexts where the underlying rings are endowed with additional structures such as grading. Unlike graded ideals, which align with the grading of the ring and often simplify the analysis, non-graded ideals lack this compatibility, thus opening avenues for more complex and nuanced investigations. These ideals often reveal hidden algebraic behaviors and influence the nature of the corresponding quotient rings.

Gorenstein rings, named in honor of the mathematician Daniel Gorenstein., are distinguished by their symmetrical and self-dual homological properties. Originally introduced in commutative algebra, Gorenstein rings have since become central objects of study due to their applications

in algebraic geometry, representation theory, and singularity theory. The Gorenstein condition on a quotient ring constructed via an ideal of the parent ring imposes stringent homological constraints that reflect deep underlying symmetries.[12]

A major class of quotient rings that attract extensive study is the family of Gorenstein rings. Informally, Gorenstein rings are characterized by a very high degree of structural symmetry. From the perspective of duality: they generalize regular local rings and exhibit striking homological behavior. In favorable settings, a Gorenstein quotient behaves like a ring whose canonical module is particularly simple, leading to duality phenomena in Ext and local cohomology, as well as strong constraints on minimal free resolutions.[13]

In the context of quotient rings formed by non-graded ideals, understanding when these rings are Gorenstein is a problem of significant interest. Such analysis involves characterizing the ideals that produce Gorenstein quotients and examining how properties of the original ring

2. Preliminaries and Definitions

Definitions 2.1 . A ring is a non-empty set R together with two binary operations, addition (+) and multiplication ($\cdot$), defined on R , such that the following axioms are satisfied:

1. $(R, +)$ is an abelian group.
2. Multiplication is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$.
3. Multiplication is distributive over addition. For all $a, b, c \in R$:
 - Left distributive law: $a \cdot (b + c) = a \cdot b + a \cdot c$
 - Right distributive law: $(a + b) \cdot c = a \cdot c + b \cdot c$

Definitions 2.2 . Let R be a (commutative) ring. A subset $I \subseteq R$ is called an **ideal** if:

1. I is an additive subgroup of $(R, +)$, and
2. for all $r \in R$ and $a \in I$, we have $ra \in I$. Since R is commutative, this also gives $ar \in I$. Symbolically, we write this as $RI \subseteq I$.

Ideals are the building blocks for constructing quotient rings R/I .

For example, for every $d \in \mathbb{Z}$, the cyclic subgroup $\langle d \rangle = d\mathbb{Z}$ generated by d is an ideal in $\mathbb{Z}$. A similar statement holds for the cyclic subgroup $\langle d \rangle$ generated by d in $\mathbb{Z}_n$. However, for a general ring R and an element $a \in R$, the cyclic subgroup $\langle a \rangle$ is almost never an ideal. For instance, in $\mathbb{Z}[x]$, the subgroup $\langle x \rangle = \{nx : n \in \mathbb{Z}\}$ is not an ideal, because $x \cdot x = x^2 \notin \langle x \rangle$.

And, for all $d \in \mathbb{Z}$, the cyclic subgroup $\langle d \rangle$ generated by d is an ideal in $\mathbb{Z}$. A similar statement holds for the cyclic subgroup $\langle d \rangle$ generated by d in $\mathbb{Z}/n\mathbb{Z}$. However, for a general ring R and an element $r \in R$, the cyclic subgroup $\langle r \rangle = \{nr : n \in \mathbb{Z}\}$ is almost never an ideal

3. grading, and Gorenstein rings

Commutative algebra is the study of commutative rings and their ideals, providing the algebraic foundation for algebraic geometry and number theory. Ideals, graded rings, and Gorenstein rings represent a progression from foundational structures to specialized, highly symmetric structures, often used to study geometric shapes and their singularities.

At the core of ring theory lies the ideal, a subset of a ring that generalizes the properties of multiples of an integer. In an academic context, we examine how ideals particularly prime and maximal ideals define the "points" of an algebraic space.

The complexity of a ring is often measured by its dimension theory and how its ideals are generated. For instance, the height of an ideal corresponds to the codimension of the subvariety it defines, establishing a direct link between algebraic properties and geometric intuition.

Definitions 3.1 graded ring is a ring such that the underlying additive group is a direct sum of abelian groups. Let R be a ring and $R = \bigoplus_{i \in \mathbb{Z}} R_i$ the direct sum of abelian groups. R is graded if $R_i R_j \subseteq R_{i+j} \forall i, j \in \mathbb{Z}$, the simple example is that of polynomial rings where R_i consists of all degree polynomial of degree i .

Definitions 3.2 If $K = \bigoplus_{i \in \mathbb{Z}} K_i$ be an module if $R_i K_j \subseteq K_{i+j} \forall i, j \in \mathbb{Z}$ then K is graded R module, K_j is called the j^{th} graded homogeneous component of K and elements of K_j are called j^{th} forms, the simple example $K[x] = \bigoplus_{i \in \mathbb{Z}} Kx^i$ where $Kx^i = 0$ if $i < 0$, then $K[x] = 0 \oplus \dots \oplus 0 \oplus k \oplus kx \oplus kx^2 \oplus \dots$

Definitions 3.3 If $I \subseteq R$ is called graded ideal if I can be generated by homogeneous elements in this the quotient ring R/I inherits natural grading

$$R/I = \bigoplus_{i \geq 0} (R_i + I)/I$$

Graded rings and ideals are essential in:

Constructing and studying projective varieties, through homogeneous coordinate rings, and are summarized in: In projective geometry, we work with homogeneous coordinates $[x_1 : x_2 : \dots : x_n]$ for a polynomial to define a well-defined set of points in projective space, it must be homogeneous (all terms have the same total degree)

Example 3.1 If the twisted cubic curve in p^3 its homogeneous ideal $I \subset K[a, b, c, d]$ is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} a & b & c \\ b & c & d \end{pmatrix}$$

The resulting ideal $I = (ac - b^2, ad - bc, bd - c^2)$ is graded because every generator is of degree 2

2. Analyzing Hilbert functions and syzygies

the Hilbert function $H_M(d)$ of a graded module M measures the dimension of its pieces as a vector space: $H_M(d) = \dim_k(M_d)$. For large d , this becomes a polynomial (the Hilbert polynomial) which carries geometric data like dimension and degree.

Syzygies are the relations between the generators of an ideal. If we have generators $f_1, f_2, f_3, \dots, f_n$ a syzygy is a vector of polynomials $(g_1, g_2, \dots, g_n)$ such that $\sum g_i f_i = 0$.

Example 3.2 if $I = (x^2, xy) \subset k[x, y]$ and $f = x^2, g = xy$ then

i) A syzygy exists because $y(x^2) - x(xy)$

ii) The syzygy is the vector $(y, -x)$

Example 3.3 let $A = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset p^4$

The coordinate ring is: $R = k[x_0, x_1, x_2, x_3, x_4]/(f)$, $\deg(x_i) = 1, \deg(f) = 5$

This ring is graded, and R is Gorenstein, reflecting the Calabi–Yau property (trivial canonical class). The study of graded modules over R especially via derived categories and matrix factorizations, plays a key role in:

1. Homological mirror symmetry (Kontsevich’s conjecture),
2. B-brane categories in physics,
3. Stability conditions and wall-crossing.

Definition 3.4 [9] (Injective Dimension). If R -module M , the injective dimension $\text{id}_R(M)$ is the length of the shortest injective resolution of M .

Theorem 3.1 [10] (Homological Depth Formula)

For a Noetherian ring R and a finitely generated R -module M of finite projective dimension, the Auslander–Buchsbaum formula states:

$$\text{pd}(M) + \text{depth}(M) = \text{depth}(R),$$

when R is Cohen–Macaulay (in fact, it holds for any Noetherian local ring R), where $\text{pd}(M)$ is the projective dimension of M .

4. Non-Graded Ideals in Gorenstein Rings

In commutative algebra, Gorenstein rings form a central class of rings that generalize regular local rings and provide a powerful framework for duality. Their defining feature is a strong form of “self-duality,” often expressed in terms of the canonical module (in the graded setting) or injective dimension / Ext-duality (in the local/ungraded setting). This duality makes Gorenstein rings especially important for studying the structure of modules, resolutions, and invariants such as Hilbert functions and multiplicities.

While much of the classical theory is developed for graded situations where rings decompose by degrees and multiplication respects that grading many naturally arising ideals are not compatible with the grading. Such ideals are called non-graded ideals (or simply ideals not generated by homogeneous elements). Even when the ambient ring is graded and Gorenstein, quotienting by a non-graded ideal can significantly change the landscape:

- The quotient ring may lose the natural grading inherited from the original ring, so graded duality statements may no longer apply directly.
- The quotient can still be Gorenstein as an ungraded ring, but its behavior as a graded Gorenstein object may differ because canonical modules may no longer match a pure degree shift $R(a)$.
- Non-graded ideals can therefore reveal subtle distinctions between “Gorenstein in the ring-theoretic sense” and “graded Gorenstein in the sense of graded canonical modules.”

This makes the study of non-graded ideals in Gorenstein rings both technically challenging and conceptually illuminating. It helps clarify exactly which properties of Gorensteinness depend on grading and which persist under more general quotient operations. In many applications—such as constructing Artinian Gorenstein algebras, analyzing deformations, and studying linkage—the choice of ideal’s grading compatibility becomes an essential part of the analysis.

In the sections that follow, we explore how non-graded ideals affect quotient structures in Gorenstein rings, what changes (and what remains stable), and how examples illustrate the difference between graded and ungraded duality behavior.

Definition 4.1 A ring R is called graded if it admits a decomposition into additive subgroups

$$R = \bigoplus_{n \geq 0} R_n$$

such that $R_m \cdot R_n \subseteq R_{m+n}$ for all $m, n \geq 0$.

A ring is non-graded (in this context) if it has no such decomposition respected by multiplication, or if we are working without keeping track of degrees

Example 4.1 (non-graded):

Any ring viewed “just as a ring” can be non-graded for the purpose of a discussion. For instance, $k[x]/(x^2)$ can be studied without using its grading, even though it does admit a natural grading.

Theorem 4.1 [5] Let $(R, \mathfrak{m})$ be a Gorenstein local ring and $I \subset R$ an ideal (no grading is assumed), with $g = \text{grade}(I) = \text{height}(I)$. Then:

(Sufficient condition.) If I is generated by an R -regular sequence $x_1, \dots, x_g$, then R/I is Gorenstein.

(Criterion, in general.) R/I is Gorenstein if and only if R/I is Cohen–Macaulay and $\text{Ext}_R^i(R/I, R) \cong R/I$.

Proof ∴. The first condition ensures R/I is Cohen–Macaulay. The second follows from the injective dimension formula:

$$\text{id}_R(R/I) \leq \text{id}_R(R) + 1 = \dim(R) + 1$$

Equality in the Gorenstein condition requires careful analysis of the ext modules.

Corollary 4.2 If I is generated by a regular sequence in a Gorenstein ring R , then R/I is Gorenstein.

Proposition 4.3 if Let R be Gorenstein and I a non-graded ideal. Then R/I is Gorenstein if and only if:

$$\text{Ext}_R^i(R/I, R) \cong \begin{cases} 0 & \text{if } i \neq \text{height}(I) \\ x & \text{if } i = \text{height}(I) \end{cases}$$

where x is a rank-one free R/I -module (the canonical module).

Remark 4.4 The structure of the canonical module is essential for understanding dualization properties in non-graded settings.

Example 4.2 Let $R = k[x_1, \dots, x_n]$ (Gorenstein) and consider the non-graded ideal:

$$I = (x_1 + x_2, x_3x_4 - 1)$$

Under suitable conditions (e.g., if the generators form a regular sequence), R/I remains Gorenstein.

Converse of the previous example. Not all non-graded ideals in Gorenstein rings yield Gorenstein quotients.

Consider:

$$R = k[x, y, z], I = (x^2, xy, xz, yz)$$

The quotient R/I is Cohen–Macaulay but not Gorenstein (the canonical module has torsion).

Definition 4.2 (Canonical Module). Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring of dimension d . A finitely generated R -module ω_R is called a canonical module of R if

$$\omega_R \otimes_R \hat{R} \cong \text{Hom}_R(H^d_{\mathfrak{m}}(R), E_R(k)),$$

where $E_R(k)$ is the injective hull of the residue field $k = R/\mathfrak{m}$ and $H^d_{\mathfrak{m}}(R)$ is the d -th local cohomology module of R with support in $\mathfrak{m}$. When R is complete, this reads

$$\omega_R = \text{Ext}_R^{d-\text{depth}(R)}(R, E)$$

Theorem 4.2 [11] (Duality for Gorenstein Rings). If R is Gorenstein, then $\omega_R \cong R$ (up to a shift in degree for graded rings).

Theorem 4.3 [5] (Canonical Module of Quotient). Let R be Gorenstein and I a non-graded ideal generated by a regular sequence. Then:

$$\omega_{R/I} \cong \text{Ext}_R^{|I|}(R/I, \omega_R) \cong \text{Ext}_R^{|I|}(R/I, R)$$

Theorem 4.5 . Let R be Gorenstein, I non-graded, and $\phi: R \rightarrow S$ a flat homomorphism. If R/I is Gorenstein, then S/IS is Gorenstein.

Proof. Flatness preserves exact sequences. The injective dimension formula remains valid under flat base change.

If R is a Gorenstein ring, $\phi: R \rightarrow S$ is a flat homomorphism and R/I is a Gorenstein ring.

$$S/IS \cong (R/I) \otimes_R S$$

A key result in algebraic geometry and commutative algebra states that Gorenstein properties (as well as Cohen–Macaulay and regular properties) are preserved under flat maps. Since R/I is Gorenstein and ϕ is flat, S/IS inherits the Gorenstein property.

The Gorenstein property can be defined via the finite injective dimension of the ring over itself. Specifically, R/I is Gorenstein if $\text{id}_R\left(\frac{R}{I}\right) < \infty$. Because S is flat over R , the functor preserves exact sequences and,

consequently, preserves the finiteness of injective dimensions. The injective dimension formula remains valid under flat base change

$$\text{id}_S\left(\frac{S}{IS}\right) = \text{id}_R\left(\frac{R}{I}\right) < \infty$$

Therefore, S/IS is Gorenstein.

The following example illustrates a graded Gorenstein ring and a graded ideal

Example 4.3 Let $R = k[x, y]$ with the standard grading:

$R = \bigoplus_{d \geq 0} R_d$, R_d = "homogeneous polynomials of degree d ".

This R is a polynomial ring, hence regular, hence Gorenstein.

Consider the graded ideal

$$I = (x^2, y^2).$$

Then I is graded because x^2 and y^2 are homogeneous.

Now the quotient

$$A = R/I$$

inherits a grading. In fact, A is an Artinian graded complete intersection (two generators, regular sequence), hence graded Gorenstein.

This example illustrates same ambient ring, but a non-graded ideal

Example 4.4 Keep $R = k[x, y]$ graded as above. Consider the ideal

$$J = (x^2 + y, y^2).$$

Here $x^2 + y$ is not homogeneous, so J is generally not graded.

Then

$$B = R/J$$

is still a quotient ring, but there is no natural inherited grading from R unless you can show J equals some graded ideal

This shows that B may still be Gorenstein as an ungraded ring (depending on algebraic properties like being a complete intersection). But it may not be graded Gorenstein in the standard sense, because B does not come with a canonical grading compatible with the quotient construction. And the example shows very simple non-graded ideal inside a graded ring

Example 4.5 Let $R = k[x]$ with grading by degree. Consider

$$J = (x + 1).$$

This is an ideal generated by a non-homogeneous element ($x + 1$ has degrees 1 and 0 terms).

Then

$$R/J \cong k$$

This shows R/J is trivially Gorenstein. But the quotient does not come from a grading induced by a graded ideal (since J is not graded in the given grading). So this illustrates:

Gorenstein behavior of R/J can survive, but graded structures may disappear.

5. Applications

Singularity Theory. Non-graded ideals arise naturally in the study of isolated singularities. Understanding Gorenstein quotients helps classify singularities via Hilbert-Kunz multiplicity.

In algebraic geometry commutative algebra, you often start with a graded ring $R = \bigoplus_{n \geq 0} R_n$ coming from a projective/affine cone. Many natural objects (like homogeneous coordinate rings) are graded.

But when you zoom in on an isolated singularity, you frequently need ideals that describe:

the tangent cone but also higher-order structure, or local equations near the singular point that are not globally homogeneous.

Those local defining equations typically are not homogeneous as polynomials, so the corresponding ideals in the local affine ring are not graded (or at least not generated by homogeneous elements).

A singularity might “look homogeneous” at first order, but the actual local ring often contains lower-order terms. Those terms force ideals to be non-graded

This example illustrates a non-graded ideal coming from a singularity

Example 5.1 take the graded ring

$$R = k[x, y]$$

with grading by total degree.

Consider the ideal generated by a polynomial with a non-homogeneous term:

$$I = (x^2 + y).$$

I is not generated by homogeneous elements (unless y is treated in a way that makes it homogeneous with the same degree—which it isn’t here).

Geometrically, the curve $x^2 + y = 0$ is smooth in the plane, but in many singularity setups you localize at a point or embed into a higher-dimensional singular space where such terms become part of the local structure.

To make the isolated singularity aspect explicit, put this inside a hypersurface singularity.

If

$$A = k[u, v, w]/(uv - w^2),$$

which has an isolated singularity at the origin (the A_1 -type surface singularity).

Inside A , consider an ideal obtained by mapping a non-homogeneous expression (for instance, by substituting variables so that lower-order terms appear). Then the corresponding ideal in A will generally fail to be graded even if the ambient presentation is graded.

If R is a local (or graded) ring and you form a quotient

$$R \rightarrow R/I,$$

then R/I is Gorenstein if it has “self-duality” properties in its local cohomology Ext groups.

Gorenstein rings are “nice” singularities algebraically, they often behave like the best possible Cohen–Macaulay type, and they have strong structural restrictions.

So many classification strategies reduce questions about a complicated ring to studying quotients that are Gorenstein, because those quotients are more rigid.

Hilbert-Kohns plurality (positive property) is applied over a field of characteristic $p > 0$, for an R module M one studies lengths like

$$\ell_R(M/\mathfrak{m}^{[q]}M) \text{ where } q = p^e,$$

with $\mathfrak{m}^{[q]}$ the Frobenius power. The growth rate is governed by Hilbert–Kunz multiplicity.

If you can find a surjection

$$R \rightarrow R/I$$

where R/I is Gorenstein, then:

1. Invariants like Hilbert–Kunz multiplicity can become easier to compute,
2. And the resulting values often correlate with the “type” of singularity (especially in families where Frobenius behavior is constrained).

Passing to Gorenstein quotients turns a classification problem into one about rigid algebraic structure, where the key invariants (such as the Hilbert function, the h-vector, and the Cohen–Macaulay type)

One clean, concrete link example (standard computation behavior)

A classic class where these ideas are used: 1-dimensional hypersurface singularities and their Artinian Gorenstein quotients (obtained by modding out by a parameter).

For instance, let

$$R = k[[t^2, t^3]] \subset k[[t]]$$

(an analytically irreducible cusp-type singularity; it’s a 1D Cohen–Macaulay local domain).

If you take a parameter x (a nonzerodivisor) and form the quotient $R/(x)$, that quotient is 0-dimensional Gorenstein (in many standard 1D CM settings, quotient by a parameter yields an Artinian Gorenstein ring).

Then Hilbert–Kunz multiplicity computations for such rings often reduce to length computations in an Artinian Gorenstein algebra, where the structure is more controllable.

Example 5.2 Let k be a field of characteristic $p > 0$ (take any prime p ; we’ll keep it symbolic).

Let

$$R = k[x, y]$$

with the usual grading $\deg x = \deg y = 1$.

Now we Choose a specific non-graded ideal I

Define the ideal

$$I = (x^2, y^2 + x).$$

I is non-graded because x^2 is homogeneous of degree 2. But $y^2 + x$ is not homogeneous (it has terms of degree 2 and degree 1).

So I is generated by a non-homogeneous element, hence I is not a graded ideal.

Form the quotient $A = R/I$

Let

$$A = R/I = k[x, y]/(x^2, y^2 + x).$$

This ring is 0-dimensional because $x^2 = 0$ kills powers of x , the relation $y^2 = -x$ expresses y^2 in terms of x , hence all sufficiently high-degree monomials vanish. Using the relations in A , $x^2 = 0$, $y^2 = -x$.

We can reduce any monomial $x^a y^b$ to a linear combination of

$$\{1, x, y, xy\}.$$

Any x^a with $a \geq 2$ vanishes. Any y^b with $b \geq 2$ can be reduced: $y^2 = -x$, so $y^b = y^{b-2} y^2 = y^{b-2}(-x)$, etc., and eventually lands in the span of $1, x, y, xy$.

Check independence (quick argument):

If $a \cdot 1 + bx + cy + d(xy) = 0$ in A , then viewing in the quotient won’t allow cancellation between different residue classes because the only relations are $x^2 = 0$ and $y^2 = -x$, which don’t create a linear dependence among those four monomials in degree ≤ 1

Therefore

$$\ell_R(A) = \dim_k(A) = 4.$$

Now Show A is Gorenstein

For a 0-dimensional local Artinian ring, “Gorenstein” is equivalent to having a 1-dimensional socle:

$$\text{Soc}(A) = \{a \in A : \mathfrak{m}a = 0\}$$

where $\mathfrak{m} = (\bar{x}, \bar{y})$ is the maximal ideal.

In our ring, compute the socle

Since $x^2 = 0$, we have $\bar{x} \cdot \bar{x} = 0$. using $y^2 = -x$, we get $\bar{y}^2 = -\bar{x}$.

Multiply xy by x and by y :

$$\begin{aligned} \bar{x} \cdot (\bar{x}\bar{y}) &= \bar{x}^2 \bar{y} = 0, \\ \bar{y} \cdot (\bar{x}\bar{y}) &= \bar{x} \bar{y}^2 = \bar{x}(-\bar{x}) = -\bar{x}^2 = 0. \end{aligned}$$

So $\bar{x}\bar{y} \in \text{Soc}(A)$.

Now check that nothing else is in the socle:

1. $\bar{x}$ is not in the socle because $\bar{y} \cdot \bar{x} = \bar{x}\bar{y} \neq 0$.
2. $\bar{y}$ is not in the socle because $\bar{x} \cdot \bar{y} = \bar{x}\bar{y} \neq 0$.
3. 1 is never in the socle.

Thus the socle is exactly 1-dimensional:

$$\text{Soc}(A) = k \cdot \bar{x}\bar{y}.$$

Therefore A is Gorenstein.

Compute the Hilbert–Kunz multiplicity $e_{\text{HK}}(A)$

Hilbert–Kunz multiplicity is defined (for $d = \dim A$) by

$$e_{\text{HK}}(A) = \lim_{e \rightarrow \infty} \frac{1}{q^d} \ell_A(A/\mathfrak{m}^{[q]}), q = p^e,$$

where $\mathfrak{m}^{[q]}$ is the Frobenius power of $\mathfrak{m}$.

Here A is 0-dimensional, so $d = 0$ and $q^d = 1$. Moreover, for any $q \geq 1$,

$\mathfrak{m}^{[q]}$ is an $\mathfrak{m}$ -primary ideal, and since A is Artinian, eventually $\mathfrak{m}^{[q]} = 0$.

But we don't even need "eventually": in dimension 0, the limit simplifies to

$$e_{\text{HK}}(A) = \ell_A(A).$$

6. Conclusion

This paper establishes a framework for the existence and classification of non-graded ideals in commutative rings whose quotient rings are Gorenstein. Extending the classical perspective on graded Gorenstein quotients, we show that the non-graded setting introduces genuinely new algebraic behavior, reflected in both the structure of the ideals and the geometry encoded by the quotient rings. and describe how these outcomes depend on the algebraic properties of the ambient ring. Overall, our results clarify the extent to which Gorenstein quotients constrain non-graded deformations, and they demonstrate that the classification problem admits sharply different regimes under appropriate hypotheses

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